SOME SIMPLE MODULES FOR THE RESTRICTED CARTAN-TYPE LIE ALGEBRAS

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Abstract. Every simple module having character height at most one for a restricted Cartan-type Lie algebra $\mathfrak{g}$ can be realized as a quotient of a module obtained by starting with a simple module S for the homogeneous component of degree zero in the natural grading of $\mathfrak{g}$, extending the action trivially to positive components and inducing up to $\mathfrak{g}$. It is shown that if S is not restricted, or if it is restricted and its maximal vector does not have exceptional weight, then the induced module is already simple.

0. Introduction

For a restricted Cartan-type Lie algebra $\mathfrak{g}$, the *restricted* simple modules have been determined in the sense that their isomorphism classes have been parametrized, concrete realizations of them have been constructed, and their dimensions have been computed (see [Sh, H1–4]). This determination is actually only modulo the same information for the restricted simple modules for the homogeneous component of $\mathfrak{g}$ of degree zero, which is reductive, and some work has yet to be done for the hamiltonian and contact algebras when the characteristic of the field is small. Now to each simple $\mathfrak{g}$-module, there corresponds a linear functional on $\mathfrak{g}$ called a character, the zero character being the one corresponding to each restricted simple module. Therefore, the next task is to study the simple modules having nonzero characters.

In 1941, Chang [C] worked with the smallest $\mathfrak{g}$, namely, the Witt algebra $W(1, \mathbf{1})$ and succeeded in determining, in the above sense, all the simple modules (arbitrary character). Later, Strade [St] gave proofs of many of Chang's results using more sophisticated methods. Koreshkov [K] studied the next smallest Witt algebra, $W(2, \mathbf{1})$, and proved many things about the simple modules, but his results are not as explicit or complete as Chang's.

Recently, the first author, working with the general Witt algebra $W(n, \mathbf{1})$, determined those simple modules having characters with height at most one [H5]. (The height of a character is the smallest degree for which the character vanishes on the corresponding filter component.) It is natural to consider these modules collectively because they are

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each a quotient of a module $Z^\chi(S)$ obtained by starting with a simple module S for the homogeneous component $\mathfrak{g}_0$ of degree zero, extending the action trivially to the positive components of $\mathfrak{g}$, and then inducing up (see 1.2 below). The trivial extension of the action is the step that requires the character height to be at most one. The induced modules $Z^\chi(S)$ played an important role in the determination of the restricted simple modules, and methods of [H5] are modeled after and partly generalize those for the restricted case.

The findings for the Witt algebra $W(n, \mathbf{1})$ show that the induced modules $Z^\chi(S)$ are simple whenever S is not restricted, or when S is restricted and its maximal vector has nonexceptional weight. The exceptional weights, defined carefully in Section 2, are the ones that appeared in the study of the restricted case. They are $n+1$ in number, as compared to p^n total weights, where p is the characteristic of the underlying field. Therefore, roughly speaking, the induced modules $Z^\chi(S)$ are usually simple. In this paper, we show that the same statement about the simplicity of the induced modules $Z^\chi(S)$ holds for any Cartan-type Lie algebra, with the corresponding exceptional weights depending on the algebra, but always small in number.

In [H5], the simple quotients of the induced modules $Z^\chi(S)$ for the Witt algebra, in the case of χ having height zero and S having maximal vector of exceptional weight, were determined by first realizing the induced modules as the terms of a certain χ-version of the usual de Rham sequence. This generalized Shen's approach in [Sh] for the restricted case. The second author has carried out in [Z] a similar determination for the special, hamiltonian, and contact algebras (assuming, when the algebra is hamiltonian or contact, that $p > r$ in the notation of Section 3 below). This, together with the results of the present paper and the earlier results mentioned above, completes the determination of the simple modules having character height at most one for the restricted Cartan-type Lie algebras (again, modulo classical information and the case of small p).

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1. Notation and Method

Let F be an algebraically closed field of characteristic $p \geq 5$ and let $\mathfrak{g}$ be a simple restricted Cartan-type Lie algebra over F. Thus $\mathfrak{g}$ belongs to one of four classes of algebras: Witt, special, hamiltonian, contact. Each of these classes will be described in detail later in the paper.

Let $\chi \in \mathfrak{g}^* = \mathrm{Hom}_F(\mathfrak{g}, F)$. A (finite-dimensional, left) $\mathfrak{g}$-module M *has character* χ provided

$$x^p \cdot m - x^{[p]} \cdot m = \chi(x)^p m$$

for all $x \in \mathfrak{g}$, $m \in M$, where x^p denotes the pth power of x in the universal enveloping algebra of $\mathfrak{g}$ and $x \mapsto x^{[p]}$ is the p-mapping defined on $\mathfrak{g}$. Not every module has a character, but at least every simple module has one [SF, Theorem 2.5, p. 207].

Generalizing the construction of the restricted enveloping algebra $u(\mathfrak{g})$ of $\mathfrak{g}$, one defines the χ-*reduced universal enveloping algebra* of $\mathfrak{g}$, denoted $u(\mathfrak{g},\chi)$, by forming the quotient of the universal enveloping algebra of $\mathfrak{g}$ by the ideal generated by $\{x^p - x^{[p]} - \chi(x)^p 1_F \mid x \in \mathfrak{g}\}$. Note that $u(\mathfrak{g},0) = u(\mathfrak{g})$. Just like with $u(\mathfrak{g})$, the vector space $u(\mathfrak{g},\chi)$ has a PBW-type basis. The $u(\mathfrak{g},\chi)$-modules are precisely the $\mathfrak{g}$-modules having character χ.

Let $\mathfrak{a}$ be a restricted subalgebra of $\mathfrak{g}$. Then χ restricts to an element of $\mathfrak{a}^*$ which we continue to denote by χ. The algebra $u(\mathfrak{a},\chi)$ identifies with a subalgebra of $u(\mathfrak{g},\chi)$ in the natural way. (See [SF, Section 5.3] for more details.)

The algebra $\mathfrak{g}$ is finite dimensional over F and possesses a natural restricted grading: $\mathfrak{g} = \dot{\sum}_i \mathfrak{g}_i$ with $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$, $\mathfrak{g}_i^{[p]} \subseteq \mathfrak{g}_{pi}$. The subspace $\mathfrak{g}_0$ of $\mathfrak{g}$ is clearly a restricted subalgebra. It has a triangular decomposition $\mathfrak{g}_0 = \mathfrak{n}_0^- \dot{+} \mathfrak{h} \dot{+} \mathfrak{n}_0$ with $\mathfrak{h}$ a maximal torus of $\mathfrak{g}_0$ and with $\mathfrak{n}_0$ (respectively, $\mathfrak{n}_0^-$) a p-nilpotent ideal of $\mathfrak{h} + \mathfrak{n}_0$ (respectively, $\mathfrak{h} + \mathfrak{n}_0^-$). For each $i \in \mathbb{Z}$, we put $\mathfrak{g}^i = \sum_{j \geq i} \mathfrak{g}_j$ and define $\mathfrak{n} = \mathfrak{n}_0 + \mathfrak{g}^1$, $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$.

The space $\mathfrak{h}$ has a basis $\{h_1, \ldots, h_d\}$ the elements of which satisfy $h_i^{[p]} = h_i$ $(1 \leq i \leq d)$. Let M be a $\mathfrak{b}$-module and let $\lambda \in F^d$. We set $M_\lambda = \{m \in M \mid h_i \cdot m = \lambda_i m$ for all $1 \leq i \leq d\}$. An element of M_λ is a *weight vector* (*of weight* λ). A nonzero $m \in M_\lambda$ is a *maximal vector* (*of weight* λ) provided $\mathfrak{n} \cdot m = 0$.

Now suppose M has character χ and let $0 \neq m \in M_\lambda$. Then, since $h_i^{[p]} = h_i$, we have $\lambda_i^p m - \lambda_i m = h_i^p \cdot m - h_i \cdot m = \chi(h_i)^p m$ for each $1 \leq i \leq d$, implying $\lambda \in \Lambda^\chi := \{\lambda \in F^d \mid \lambda_i^p - \lambda_i = \chi(h_i)^p$ for all $1 \leq i \leq d\}$. In particular, if M has a maximal vector of weight λ, then necessarily $\lambda \in \Lambda^\chi$. Note that if $\chi(\mathfrak{h}) = 0$, then $\Lambda^\chi = \mathbb{F}_p^d =: \Lambda$, where $\mathbb{F}_p$ is the prime subfield of F.

1.1 Lemma. *Let $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{n}) = 0$ and let M be a $u(\mathfrak{g},\chi)$-module. The following conditions are equivalent:*

(1) *M is nonzero and is generated by each of its maximal vectors,*
(2) *M is simple.*

Proof. Assume (1) holds and let M' be a nonzero submodule of M. Choose a simple $\mathfrak{b}$-submodule S of M'. Now $\mathfrak{n}_0$ is a p-nilpotent ideal of $\mathfrak{h} + \mathfrak{n}_0$ and the grading on $\mathfrak{g}$ is restricted, so $\mathfrak{n}$ is a p-nilpotent ideal of $\mathfrak{b}$. Since S has character χ and $\chi(\mathfrak{n}) = 0$, it follows that for each $x \in \mathfrak{n}$, $x^{p^l} \cdot S = x^{[p]^l} \cdot S = 0$ for some $l \in \mathbb{N}$. Therefore, $\mathfrak{n} \cdot S = 0$ [SF, Corollary 3.8, p. 19]. This implies that S is simple as $\mathfrak{h}$-module. Since $\mathfrak{h}$ is abelian, S must be one-dimensional [SF, Lemma 5.6, p. 31], so $S = Fm$ for some nonzero $m \in S$. Clearly m is a maximal vector. By assumption, m generates M, so that $M' = M$. Thus (2) holds.

Since a maximal vector is nonzero by definition, the other implication is obvious. □

Since $\mathfrak{g}^1 \lhd \mathfrak{g}^0$, any $\mathfrak{g}_0$-module becomes a $\mathfrak{g}^0$-module via the canonical map $\mathfrak{g}^0 \to \mathfrak{g}^0/\mathfrak{g}^1 \cong \mathfrak{g}_0$. In particular, a $\mathfrak{g}_0$-module can in this way be viewed as a $\mathfrak{b}$-module; the notion of maximal vector applied to this situation coincides with the classical one for $\mathfrak{g}_0$-modules relative to the subalgebra $\mathfrak{h} + \mathfrak{n}_0$.

For any $u(\mathfrak{g}^0, \chi)$-module M, the *induced* $u(\mathfrak{g}, \chi)$-module $Z^\chi(M)$ is defined by

$$Z^\chi(M) = u(\mathfrak{g}, \chi) \otimes_{u(\mathfrak{g}^0,\chi)} M.$$

Following Strade [St], we define the *height* of χ by

$$\mathrm{ht}\,\chi = \min\{i \geq t \,|\, \chi(\mathfrak{g}^i) = 0\},$$

where $t = \min\{i \,|\, \mathfrak{g}_i \neq 0\}$. If $\mathrm{ht}\,\chi \leq 1$ and M is a $u(\mathfrak{g}_0, \chi)$-module, then, since $\chi(\mathfrak{g}^1) = 0$, M has character χ when viewed as a $\mathfrak{g}^0$-module as in the preceding paragraph, so that $Z^\chi(M)$ is defined. These induced modules are useful for the study of simple modules for the following reason.

1.2 Proposition. *Let $\chi \in \mathfrak{g}^*$ with $\mathrm{ht}\,\chi \leq 1$ and let M be a simple $u(\mathfrak{g}, \chi)$-module. Then M is a homomorphic image of $Z^\chi(S)$ for some simple $u(\mathfrak{g}_0, \chi)$-module S.*

Proof. M has a simple $u(\mathfrak{g}^0, \chi)$-submodule S. Now $\mathfrak{g}^1 \triangleleft \mathfrak{g}^0$, so arguing just as in the proof of 1.1, we deduce that $\mathfrak{g}^1$ acts trivially on S. This implies that S is a (simple) $u(\mathfrak{g}_0, \chi)$-module (see the discussion after 1.1). The inclusion map $S \to M$ is a $u(\mathfrak{g}^0, \chi)$-homomorphism, so it induces a $u(\mathfrak{g}, \chi)$-homomorphism $Z^\chi(S) \to M$, which is surjective since M is simple. $\square$

As pointed out in the introduction, the theory of *restricted* representations of $\mathfrak{g}$ gave rise to certain weights in Λ called *exceptional weights* ([Sh], [H1–4], [N]), the collection of which we shall denote by Λ_e. The precise definitions of these weights are given in Sections 2 and 3.

The main result of the paper (4.3) is a corollary of the next theorem.

1.3 Theorem. *Let $\chi \in \mathfrak{g}^*$ with $\mathrm{ht}\,\chi \leq 1$, let M be a $u(\mathfrak{g}_0, \chi)$-module, and let v be a maximal vector in $Z^\chi(M)$ of weight λ. If either $\chi(\mathfrak{n}_0^-) \neq 0$ or M has no maximal vector of exceptional weight, then $v = 1 \otimes m_0$ with $m_0 \in M$ a maximal vector of weight λ.*

The proof of this theorem occupies the next two sections. The case with $\mathfrak{g}$ the Witt algebra was proved in [H5]; our proof where $\mathfrak{g}$ is the special algebra requires just a few additional arguments to recover the Witt algebra case, so we go ahead and include them in order to present a somewhat unified approach.

2. The Witt and Special Algebras.

In this section, we prove 1.3 in the case $\mathfrak{g}$ is either the Witt algebra or the special algebra. We begin by describing these algebras, drawing most of the notation and standard results from [SF]. (See also [BW].)

Fix $n \in \mathbb{N}$ and let $a, b \in \mathbb{Z}^n$. We write $a \leq b$ if $a_i \leq b_i$ for all $1 \leq i \leq n$ and we write $a < b$ if $a \leq b$ but $a \neq b$. If $a, b \geq 0$, define $\binom{a}{b} = \prod_i \binom{a_i}{b_i}$, where $\binom{a_i}{b_i}$ is the usual binomial coefficient with the convention that $\binom{a_i}{b_i} = 0$ unless $b_i \leq a_i$. Set $A = A(n, \mathbf{1}) = \{a \in \mathbb{Z}^n \mid 0 \leq a \leq \tau\}$, where $\tau := (p-1, \ldots, p-1)$. The *divided power algebra* $\mathfrak{A} = \mathfrak{A}(n, \mathbf{1})$ is the associative F-algebra having F-basis $\{x^{(a)} \mid a \in A\}$ and multiplication subject to the rule

$$x^{(a)}x^{(b)} = \binom{a+b}{a} x^{(a+b)},$$

where $x^{(c)} := 0$ if $c \notin A$.

For each $1 \leq i \leq n$, let D_i denote the derivation of $\mathfrak{A}$ uniquely determined by the property $D_i x^{(a)} = x^{(a-\epsilon_i)}$, where ϵ_i is the n-tuple with jth entry δ_{ij} (= Kronecker delta). Then the *Witt algebra* $W = W(n, \mathbf{1})$ is the restricted Lie algebra $\operatorname{Der}_F \mathfrak{A} = \sum_i \mathfrak{A} D_i$, which has F-basis $\{x^{(a)} D_i \mid a \in A, 1 \leq i \leq n\}$. The bracket product in W satisfies

$$[x^{(a)}D_i, x^{(b)}D_j] = \binom{a+b-\epsilon_i}{a} x^{(a+b-\epsilon_i)} D_j - \binom{a+b-\epsilon_j}{b} x^{(a+b-\epsilon_j)} D_i,$$

and the p-mapping is p-fold composition: $D^{[p]} := D^p$ $(D \in W)$. Putting $x_i = x^{(\epsilon_i)}$, we have $(x_i D_i)^{[p]} = x_i D_i$ and $(x^{(a)} D_i)^{[p]} = 0$ if $a \neq \epsilon_i$ $(1 \leq i \leq n)$.

Given $a \in \mathbb{Z}^n$, set $|a| = \sum_i a_i$. Defining $\mathfrak{A}_k = \langle x^{(a)} \mid a \in A, |a| = k \rangle$ and $W_k = \sum_j \mathfrak{A}_{k+1} D_j$ we have $W = \dot{\sum}_{i=-1}^{s} W_i$, where $s = n(p-1) - 1$. This is the restricted grading on W referred to in Section 1. The restricted subalgebra W_0 of W is isomorphic to $\mathfrak{gl}(\mathfrak{A}_1)$ via $D \mapsto D|_{\mathfrak{A}_1}$ $(D \in W_0)$. Composing this isomorphism with the isomorphism $\mathfrak{gl}(\mathfrak{A}_1) \to \mathfrak{gl}_n(F)$ obtained by identifying $x_i \in \mathfrak{A}_1$ with the n-dimensional column vector having jth entry δ_{ij}, we obtain an isomorphism $W_0 \to \mathfrak{gl}_n(F)$ that sends $x_i D_j$ to e_{ij} (= $n \times n$-matrix with 1 in the (i, j)-position and zeros elsewhere).

Now suppose $n > 1$. For $1 \leq i, j \leq n$ and $x \in \mathfrak{A}$, put

$$D_{ij}(x) = D_j(x) D_i - D_i(x) D_j.$$

The *special algebra* is $S = S(n, \mathbf{1}) = \langle D_{ij}(x) \mid 1 \leq i, j \leq n, x \in \mathfrak{A} \rangle$, a restricted subalgebra of W. The restricted grading on S of Section 1 is obtained by putting $S_i = W_i \cap S$. We have $S_{-1} = W_{-1}$ and the isomorphism $W_0 \to \mathfrak{gl}_n(F)$ described above induces an isomorphism $S_0 \to \mathfrak{sl}_n(F)$.

In the next lemma and below, we use the notation $D^a := \prod_i D_i^{a_i}$ $(a \in \mathbb{Z}^n,\ a \geq 0)$.

2.1 Lemma. *Let $\chi \in W^*$. The following formulas hold in the algebra $u(W, \chi)$:*

(1) $(x^{(b)} D_i) D^a = \sum_{c \geq 0} (-1)^{|c|} \binom{a}{c} D^{a-c} (x^{(b-c)} D_i)$,
(2) $(D_{ij}(x^{(b)})) D^a = \sum_{c \geq 0} (-1)^{|c|} \binom{a}{c} D^{a-c} (D_{ij}(x^{(b-c)}))$

$(a, b \geq 0,\ 1 \leq i, j \leq n)$.

Proof. Part (1) follows from [SF, Proposition 1.3(4), p. 9], and part (2) follows from (1) by writing $D_{ij}(x^{(b)}) = x^{(b-\epsilon_j)}D_i - x^{(b-\epsilon_i)}D_j$. $\square$

For the remainder of this section, we assume $\mathfrak{g} \in \{W, S\}$. Put $h_i = x_iD_i - x_{i+1}D_{i+1}$ for $1 \leq i < n$, as well as $h_n = x_nD_n$. The triangular decomposition $\mathfrak{g}_0 = \mathfrak{n}_0^- \dot{+} \mathfrak{h} \dot{+} \mathfrak{n}_0$ of Section 1 is obtained by setting $\mathfrak{n}_0^- = \sum_{i>j} Fx_iD_j$, $\mathfrak{h} = \sum_{i=1}^d Fh_i$, $\mathfrak{n}_0 = \sum_{i<j} Fx_iD_j$, where $d = n - \delta_{\mathfrak{g}S}$.

For $0 \leq i \leq d$, set

$$\omega_i = \epsilon_i - \delta_{\mathfrak{g}W}\epsilon_n,$$

where ϵ_i denotes the d-tuple with jth entry δ_{ij} ($\epsilon_0 = 0$). Then $\Lambda_e = \{\omega_0, \omega_1, \dots, \omega_d\}$ is the set of exceptional weights.

To begin the proof of 1.3, we assume its hypothesis: Let $\chi \in \mathfrak{g}^*$ with $\operatorname{ht}\chi \leq 1$, let M be a $u(\mathfrak{g}_0, \chi)$-module, let $v \in Z^\chi(M)$ be a maximal vector of weight λ and assume that either $\chi(\mathfrak{n}_0^-) \neq 0$ or M has no maximal vector of exceptional weight.

Since $\mathfrak{g}_{-1}$ has ordered basis $\{D_1, \dots, D_n\}$, it follows from the PBW theorem that any element of $Z^\chi(M)$ can be written in the form $\sum_{a\in A} D^a \otimes m_a$ with the m_a uniquely determined elements of M. In particular, the maximal vector v can be written thus: $v = \sum_{a\in A} D^a \otimes m_a$. This meaning of m_a remains in force for the rest of the section.

For $a \in A$, define $\lambda(a) \in F^d$ by

$$\lambda(a)_i = \begin{cases} \lambda_i + a_i - a_{i+1}, & 1 \leq i < n, \\ \lambda_n + a_n, & i = n. \end{cases}$$

The next lemma says that m_a is a weight vector of weight $\lambda(a)$.

2.2 Lemma. *If $a \in A$, then $h_i \cdot m_a = \lambda(a)_i m_a$ for $1 \leq i \leq d$.*

Proof. For $1 \leq i < n$, we have from 2.1(2)

$$\begin{aligned} h_i \cdot v = D_{i,i+1}(x^{(\epsilon_i+\epsilon_{i+1})}) \cdot v &= \sum_a D_{i,i+1}(x^{(\epsilon_i+\epsilon_{i+1})})D^a \otimes m_a \\ &= \sum_a D^a D_{i,i+1}(x^{(\epsilon_i+\epsilon_{i+1})}) \otimes m_a - \sum_a a_i D^{a-\epsilon_i} D_{i,i+1}(x^{(\epsilon_{i+1})}) \otimes m_a \\ &\quad - \sum_a a_{i+1} D^{a-\epsilon_{i+1}} D_{i,i+1}(x^{(\epsilon_i)}) \otimes m_a. \end{aligned}$$

The final three sums come from the terms in 2.1(2) corresponding to $c = 0$, ϵ_i, ϵ_{i+1}, respectively, which are the only choices of c that can possibly make a nonzero contribution. Substituting $D_{i,i+1}(x^{(\epsilon_i+\epsilon_{i+1})}) = h_i$, $D_{i,i+1}(x^{(\epsilon_{i+1})}) = D_i$, $D_{i,i+1}(x^{(\epsilon_i)}) = -D_{i+1}$ and collecting terms gives

$$h_i \cdot v = \sum_a D^a \otimes (h_i - a_i + a_{i+1}) \cdot m_a.$$

On the other hand, $h_i \cdot v = \lambda_i v = \sum_a D^a \otimes \lambda_i m_a$. Combining this with the previous equation and using uniqueness of expression we get the desired formula. If $i = n$ (so that $\mathfrak{g} = W$), then a similar proof, except using 2.1(1), gives the formula. □

At present, m_a is defined only for $a \in A$. It is convenient to extend the definition as follows: If $b \in \mathbb{Z}^n$ and $b \not\leq \tau$, put $m_b = 0$; if $a \in A$ and $a_i = 0$, put $m_{a-\epsilon_i} = \chi(D_i)^p m_{a+(p-1)\epsilon_i}$. The motivation for these definitions appears in the proof of the next lemma.

2.3 Lemma. *Let $1 \leq i, j \leq n$ and let $b \in A$.*

(1) *If either $1 < t < p-1$, $i \neq j$, or $1 \leq t < p-1$, $i < j$, then*

$$\binom{b_i + t - 1}{t-1} x_i D_j \cdot m_{b+(t-1)\epsilon_i} = \binom{b_i + t}{t} m_{b+t\epsilon_i - \epsilon_j}.$$

(2) *If $i \neq j$, then*

$$(b_j + 1) x_i D_j \cdot m_{b+\epsilon_j} = (b_i + 1)[b_j - b_i/2 + x_i D_i - x_j D_j] \cdot m_{b+\epsilon_i}.$$

(3) *If $\mathfrak{g} = W$ and $1 \leq t < p-1$, then*

$$\binom{b_i + t}{t} x_i D_i \cdot m_{b+t\epsilon_i} = \binom{b_i + t}{t+1} m_{b+t\epsilon_i}.$$

Proof. (1) Assuming either of the stated conditions, we have $D_{ij}(x^{((t+1)\epsilon_i)}) = -x^{(t\epsilon_i)} D_j \in \mathfrak{n}$. Therefore, using that $\mathfrak{n} \cdot v = 0$ and then 2.1(2), we obtain

$$\begin{aligned} 0 = -D_{ij}(x^{((t+1)\epsilon_i)}) \cdot v &= -\sum_a (-1)^t \binom{a_i}{t} D^{a-t\epsilon_i} D_{ij}(x^{(\epsilon_i)}) \otimes m_a \\ &\quad - \sum_a (-1)^{t-1} \binom{a_i}{t-1} D^{a-(t-1)\epsilon_i} D_{ij}(x^{(2\epsilon_i)}) \otimes m_a \\ &= \sum_{a \in A} D^{a-t\epsilon_i+\epsilon_j} \otimes (-1)^t \binom{a_i}{t} m_a \\ &\quad - \sum_{a \in A} D^{a-(t-1)\epsilon_i} \otimes (-1)^t \binom{a_i}{t-1} x_i D_j \cdot m_a. \end{aligned} \tag{2.3.1}$$

The first two sums appearing above come from the terms $c = t\epsilon_i$ and $c = (t-1)\epsilon_i$, respectively, in 2.1(2); all other choices of c make no contribution, either because $x^{((t+1)\epsilon_i - c)} = 0$ or because $D_{ij}(x^{((t+1)\epsilon_i - c)})$ is in $\mathfrak{g}^1$ which annihilates M.

Writing $n_a = (-1)^t \binom{a_i}{t} m_a$, we have

$$\sum_{a \in A} D^{a-t\epsilon_i+\epsilon_j} \otimes n_a = \sum_{\substack{a \in A \\ a_i \geq t \\ a_j < p-1}} D^{a-t\epsilon_i+\epsilon_j} \otimes n_a + \sum_{\substack{a \in A \\ a_i \geq t \\ a_j = p-1}} D^{a-t\epsilon_i+\epsilon_j} \otimes n_a. \tag{2.3.2}$$

Now $D^{a-t\epsilon_i+\epsilon_j} = \chi(D_j)^p D^{a-t\epsilon_i-(p-1)\epsilon_j}$ whenever $a_j = p-1$, so

$$\sum_{\substack{a\in A\\ a_i\geq t\\ a_j=p-1}} D^{a-t\epsilon_i+\epsilon_j}\otimes n_a = \sum_{\substack{b\in A\\ b_i\leq p-1-t\\ b_j=0}} D^b\otimes \chi(D_j)^p n_{b+t\epsilon_i+(p-1)\epsilon_j} = \sum_{\substack{b\in A\\ b_j=0}} D^b\otimes n_{b+t\epsilon_i-\epsilon_j},$$

where we have used the conventions established before the statement of the lemma. A more elementary index shift in the first sum on the right of (2.3.2) expresses it as $\sum_{\substack{b\in A\\ b_j\neq 0}} D^b\otimes n_{b+t\epsilon_i-\epsilon_j}$, so

$$\sum_{a\in A} D^{a-t\epsilon_i+\epsilon_j}\otimes n_a = \sum_{b\in A} D^b\otimes n_{b+t\epsilon_i-\epsilon_j}.$$

After performing a similar index shift on the last sum in (2.3.1) and combining with the first sum as rewritten above, (2.3.1) becomes

$$0 = \sum_{b\in A} D^b\otimes\left[(-1)^t\binom{b_i+t}{t} m_{b+t\epsilon_i-\epsilon_j} - (-1)^t\binom{b_i+t-1}{t-1} x_iD_j\cdot m_{b+(t-1)\epsilon_i}\right],$$

so the result follows.

(2) Assume $i\neq j$. Since $D_{ij}(x^{(2\epsilon_i+\epsilon_j)}) = x^{(2\epsilon_i)}D_i - x^{(\epsilon_i+\epsilon_j)}D_j$ is in $\mathfrak{n}$, it annihilates v. One argues as in the proof of (1) to obtain the result.

(3) Assume $\mathfrak{g} = W$ and $1\leq t < p-1$. Then $x^{((t+1)\epsilon_i)}D_i$ is in $\mathfrak{n}$ and hence it annihilates v. Again, one argues as in the proof of (1), except this time using 2.1(1) instead of 2.1(2). □

Now that the required formulas have been established, we are ready to carry out the main portion of the proof of 1.3, which amounts to showing that in our expression $v = \sum_a D^a\otimes m_a$, each m_a is zero unless $a = 0$. We will then have $v = 1\otimes m_0$ and it is an easy matter to show m_0 is a maximal vector. The proof proceeds in steps.

2.4 Lemma. *Let $a\in A$. If there exist $1\leq i,j\leq n$ with $i\neq j$ such that $a_j\neq p-1$ and either $a_i\geq 3$ or $a_i\geq 2$ and $i<j$, then $m_a = 0$.*

Proof. Let $1\leq i,j\leq n$ with $i\neq j$ and assume $a_j\neq p-1$. Then 2.3(1) with $b = a-t\epsilon_i+\epsilon_j$ gives

$$\binom{a_i-1}{t-1} x_iD_j\cdot m_{a-\epsilon_i+\epsilon_j} = \binom{a_i}{t} m_a,$$

which is valid for $t\in\{1,2\}$ if $i<j$ and $a_i\geq 2$, and valid for $t\in\{2,3\}$ if $a_i\geq 3$. In either case, we obtain two equations which, when solved, yield $m_a = 0$. □

2.5 Lemma. *Let $a\in A$. If $a_i = p-1$ for some $1\leq i\leq n$, then $m_a = 0$.*

Proof. Assume $a_i = p-1$ for some $1\leq i\leq n$. Since $p>3$, we have $a_i\geq 3$. Therefore, in view of 2.4, we may assume $a = \tau = (p-1,\dots,p-1)$.

Now 2.3(1) with $t = 1, b = \tau$ gives $x_iD_j \cdot m_\tau = 0$ whenever $1 \le i < j \le n$. Next, for each $1 \le i < n$, 2.3(2) with $j = i+1$, $b = \tau - \epsilon_i$ gives $h_i \cdot m_\tau = (x_iD_i - x_{i+1}D_{i+1}) \cdot m_\tau = 0$. And if $\mathfrak{g} = W$, then 2.3(3) with $t = 1$, $b = \tau - \epsilon_n$, $i = n$ gives $h_n \cdot m_\tau = -m_\tau$. So if $m_\tau \ne 0$, then m_τ is a maximal vector of exceptional weight ω_0.

Checking the hypotheses of 1.3 we see that we may assume $\chi(\mathfrak{n}_0^-) \ne 0$, that is, $\chi(x_iD_j) \ne 0$ for some $1 \le j < i \le n$. Then 2.3(1) with $t = 2$, $b = \tau - \epsilon_i$ gives $x_iD_j \cdot m_\tau = 0$. Therefore, $\chi(x_iD_j)^p m_\tau = (x_iD_j)^p \cdot m_\tau = 0$, which implies $m_\tau = 0$, as desired. □

2.6 Lemma. *Let $a \in A$. If there exist $1 \le i < j < n$ such that $a_i \ne 0$ and $a_j \ne 0$, then $m_a = 0$.*

Proof. Assume the hypothesis. By 2.5, we may assume $a_k \ne p - 1$ for all $1 \le k \le n$. Putting $b = a - \epsilon_i + \epsilon_j$ we have $b_j \ge 2$ and $b_n \ne p - 1$, whence $m_b = 0$ by 2.4. Therefore, 2.3(1) with $t = 1$ and this choice of b yields $m_a = 0$. □

2.7 Lemma. *Let $a \in A$. If $a_i \ge 2$ for some $1 \le i \le n$, then $m_a = 0$.*

Proof. Suppose $a_i \ge 2$ for some $1 \le i \le n$. We assume $m_a \ne 0$ and shall derive a contradiction. By 2.5, we have $a_n \ne p-1$. If it were the case that $i \ne n$, then we could let $j = n$ in 2.4 to get $m_a = 0$. Thus $i = n$ and $2 \le a_n < p - 1$. By replacing a if necessary, we may assume that if $c \in A$ and $c_n > a_n$, then $m_c = 0$. Let $1 \le k < n$ and suppose $a_k \ne 0$. Then 2.3(1) with i replaced by k, $j = n$, $t = 1$, $b = a - \epsilon_k + \epsilon_n$ yields $m_a = 0$ (since $b_n = a_n + 1 > a_n$, implying $m_b = 0$ by the above assumption). We conclude that $a_k = 0$ for all $1 \le k < n$.

Suppose $\mathfrak{g} = W$. Then 2.3(3) with $i = n$, $b = a - t\epsilon_n$ yields (in view of 2.2)

$$\binom{a_n}{t}\lambda(a)_n m_a = \binom{a_n}{t+1} m_a$$

for $t \in \{1, 2\}$. Since $m_a \ne 0$, the coefficients on both sides must be equal. We thus obtain two equations (one for each t), which easily lead to $a_n = -1$ in F, implying $a_n = p-1$. This contradicts an observation made earlier in the proof. Therefore, the lemma is established for the case $\mathfrak{g} = W$.

Now assume $\mathfrak{g} = S$. In particular, $n > 1$. Therefore, we can let $i = n$, $j = 1$ in 2.4 to see that $a_n = 2$, that is, $a = 2\epsilon_n$. Then 2.3(1) with $i = n$, $t = 2$, $b = \epsilon_j$ gives $(x_nD_j) \cdot m_{\epsilon_j+\epsilon_n} = m_a$ $(1 \le j < n)$, from which it follows that

$$m_{\epsilon_j+\epsilon_n} \ne 0 \qquad (1 \le j \le n) \tag{2.7.1}$$

(since $m_a \ne 0$).

By letting $t = 1$, $b = \epsilon_k + \epsilon_l$ in 2.3(1) we find that

$$x_iD_j \cdot m_{\epsilon_k+\epsilon_l} = \begin{cases} 2^{\delta_{il}} m_{\epsilon_i+\epsilon_l}, & j = k, \\ 2^{\delta_{ik}} m_{\epsilon_i+\epsilon_k}, & j = l, \\ 0, & j \notin \{k, l\}, \end{cases} \tag{2.7.2}$$

for all $1 \leq i, j, k, l \leq n$ with $i < j$. (For this, we have used the fact that if $c \in A$ and $c_j = 0$ for some $1 \leq j \leq n$, then $m_{c-\epsilon_j} = \chi(D_j)^p m_{c+(p-1)\epsilon_j}$, which is zero by 2.5.) In particular, we get

$$x_i D_j \cdot m_{\epsilon_1+\epsilon_n} = 0 \qquad (1 \leq i < j \leq n)$$

where we have used that $m_{\epsilon_i+\epsilon_1} = 0$ if $1 < i < n$ (2.6) and $m_{2\epsilon_1} = 0$ (2.4). Therefore, $m_{\epsilon_1+\epsilon_n}$ is a maximal vector (using (2.7.1)).

Let $1 \leq i < n$. By 2.3(2) with $j = i+1$, $b = \epsilon_n$, and then by (2.7.2), we have

$$h_i \cdot m_{\epsilon_i+\epsilon_n} = \left\{ \begin{array}{ll} x_i D_{i+1} \cdot m_{\epsilon_{i+1}+\epsilon_n}, & i \leq n-2 \\ 2x_{n-1} D_n \cdot m_{2\epsilon_n} - m_{\epsilon_{n-1}+\epsilon_n}, & i = n-1 \end{array} \right\} = m_{\epsilon_i+\epsilon_n}.$$

Thus, $\lambda(\epsilon_i + \epsilon_n)_i = 1$ (2.2 and (2.7.1)). Hence,

$$\lambda(\epsilon_1 + \epsilon_n)_i = \lambda_i + \delta_{i1} - \delta_{i,n-1} = \lambda(\epsilon_i + \epsilon_n)_i + \delta_{i1} - 1 = \delta_{i1}.$$

It follows that $m_{\epsilon_1+\epsilon_n}$ is a maximal vector of exceptional weight $\lambda(\epsilon_1 + \epsilon_n) = \omega_1$.

Checking the hypotheses of 1.3, we see that it must be the case that $\chi(\mathfrak{n}_0^-) \neq 0$, so that $\chi(x_i D_j) \neq 0$ for some $1 \leq j < i \leq n$. Now 2.3(1) with $t = 2$, $b = \epsilon_n$ gives $x_i D_j \cdot m_{\epsilon_i+\epsilon_n} = 0$. Arguing as in the proof of 2.5, we obtain $m_{\epsilon_i+\epsilon_n} = 0$ which is in conflict with (2.7.1). Because of this contradiction, we deduce that in fact $m_a = 0$, as desired. □

2.8. *Completion of proof of 1.3 when* $\mathfrak{g} \in \{W, S\}$.

Let $a \in A$. If $a_n = 1$ and $a_i \neq 0$ for some $1 \leq i < n$, then 2.3(1) with $j = n$, $t = 1$, $b = a - \epsilon_i + \epsilon_n$ gives $m_a = x_i D_n \cdot m_b = 0$, the last equality from 2.7 since $b_n = 2$. From this observation, together with 2.6 and 2.7, we conclude that

$$v = 1 \otimes m_0 + \sum_{i=1}^{n} D_i \otimes m_{\epsilon_i}.$$

Assume $m_{\epsilon_l} \neq 0$ for some $1 \leq l \leq n$ and further assume this l is the least such index. From 2.3(1) with $t = 1$, $b = \epsilon_k$, we get

$$x_i D_j \cdot m_{\epsilon_k} = \delta_{jk} m_{\epsilon_i} \qquad (1 \leq k \leq n) \tag{2.8.1}$$

for $1 \leq i < j \leq n$ (using also 2.5 as in the comment after (2.7.2)). It follows that m_{ϵ_l} is a maximal vector.

Next, 2.3(2) with $b = 0$ gives $x_l D_i \cdot m_{\epsilon_i} = (x_l D_l - x_i D_i) \cdot m_{\epsilon_l}$ for all $1 \leq i \leq n$ with $i \neq l$. This, together with (2.8.1), implies

$$(x_l D_l - x_i D_i) \cdot m_{\epsilon_l} = \begin{cases} m_{\epsilon_l}, & i > l, \\ 0, & i \leq l. \end{cases}$$

Now for $1 \le i < n$, we have $h_i = x_iD_i - x_{i+1}D_{i+1} = (x_lD_l - x_{i+1}D_{i+1}) - (x_lD_l - x_iD_i)$. Applying this element to m_{ϵ_l} and using the previous formula, we find that $h_i \cdot m_{\epsilon_l} = \delta_{il}m_{\epsilon_l}$, whence $\lambda(\epsilon_l)_i = \delta_{il}$ for $1 \le i < n$ (using 2.2).

Suppose for the moment that $\mathfrak{g} = W$. Then 2.3(3) with $t = 1$, $b = 0$, $i = n$ yields $h_n \cdot m_{\epsilon_n} = 0$, whence $\lambda(\epsilon_n)_n = 0$. Thus, for $j < n$ we have $\lambda(\epsilon_j)_n = \lambda_n = \lambda(\epsilon_n)_n - 1 = -1$, implying

$$\lambda(\epsilon_l)_n = \begin{cases} -1, & \text{if } l < n, \\ 0, & \text{if } l = n. \end{cases}$$

Therefore, returning to the case of arbitrary $\mathfrak{g} \in \{W, S\}$ we see that m_{ϵ_l} is a maximal vector of exceptional weight $\lambda(\epsilon_l) = \omega_l$.

Checking the hypotheses of 1.3, we see that it must be the case that $\chi(\mathfrak{n}_0^-) \neq 0$, so that $\chi(x_iD_j) \neq 0$ for some $1 \le j < i \le n$. By 2.3(1) with $t = 2$, $b = 0$, we have (using 2.5) $x_iD_j \cdot m_{\epsilon_i} = 0$, which implies $m_{\epsilon_i} = 0$ (see last paragraph of proof of 2.5). Now if $l < k \le n$, then (2.8.1) gives $x_lD_k \cdot m_{\epsilon_k} = m_{\epsilon_l} \neq 0$, whence $m_{\epsilon_k} \neq 0$. Therefore, we have $i < l$. Next,

$$x_iD_j \cdot m_{\epsilon_l} = [x_iD_l, x_lD_j] \cdot m_{\epsilon_l} = x_iD_l \cdot x_lD_j \cdot m_{\epsilon_l} - x_lD_j \cdot x_iD_l \cdot m_{\epsilon_l}.$$

Since $x_lD_j \cdot m_{\epsilon_l} = 0$ by 2.3(1) with $i = l$, $t = 2$, $b = 0$, and also $x_iD_l \cdot m_{\epsilon_l} = m_{\epsilon_i} = 0$ (2.8.1), we get $x_iD_j \cdot m_{\epsilon_l} = 0$, implying $m_{\epsilon_l} = 0$. This is contrary to our choice of l, so we conclude that $m_{\epsilon_k} = 0$ for all $1 \le k \le n$. In other words, $v = 1 \otimes m_0$.

By 2.2, m_0 is a weight vector of weight $\lambda(0) = \lambda$. Since v is a maximal vector, $v \neq 0$, so $m_0 \neq 0$ as well. Finally, 2.3(1) with $t = 1$, $b = 0$ gives $x_iD_j \cdot m_0 = 0$ whenever $1 \le i < j \le n$. Thus, m_0 is a maximal vector of weight λ. $\square$

3. The Contact and Hamiltonian Algebras.

In this section, we complete the proof of 1.3 by considering the cases of the contact and hamiltonian algebras. We begin by describing these algebras. (Again, see [SF] or [BW] for more details.)

Let $r \in \mathbb{N}$ and set $n = 2r + 1$. Define

$$\sigma(i) = \begin{cases} 1, & 0 \le i \le r, \\ -1, & r < i \le n. \end{cases}$$

For $0 < i < n$, put $i' = i + \sigma(i)r$, and also put $0' = n$ and $n' = 0$.

Let $D_K : \mathfrak{A}(n, \mathbf{1}) \to W(n, \mathbf{1})$ denote the F-linear mapping given by $D_K(f) = \sum_{i=1}^n f_iD_i$, where

$$f_i = x_iD_n(f) + \sigma(i')D_{i'}(f) \quad (1 \le i \le 2r),$$
$$f_n = 2f - \sum_{i=1}^{2r} \sigma(i)x_if_{i'}.$$

Then D_K is injective and its image is a restricted subalgebra of $W(n,\mathbf{1})$. The *contact algebra* is

$$K = K(n,\mathbf{1}) = \begin{cases} \mathfrak{A}(n,\mathbf{1}), & n+3 \not\equiv 0 \mod p, \\ \sum_{a<\tau} Fx^{(a)}, & n+3 \equiv 0 \mod p \end{cases}$$

($\tau = (p-1,\dots,p-1)$) with the Lie bracket product given by $\langle f,g\rangle = D_K^{-1}([D_K(f),D_K(g)])$ ($f,g \in K$) and with p-mapping given by $f^{[p]} = D_K^{-1}([D_K(f)]^p)$ ($f \in K$). We will require only some special cases of the bracket product, which we gather together in the next lemma.

It is convenient to define $\epsilon_0 = 0 \in A$, so that $x_0 = x^{(\epsilon_0)} = x^{(0)} \in K$. Put $h_i = x^{(\epsilon_i+\epsilon_{i'})}$ for $1 \le i \le n$. For $a \in A(n,\mathbf{1})$, put $\|a\| = |a| + a_n - 2$ (where $|a| = \sum_{i=1}^n a_i$).

3.1 Lemma [SF, Proposition 5.3 and p. 173]**.** *Let $a \in A(n,\mathbf{1})$.*

(1) $\langle x_i, x^{(a)}\rangle = \sigma(i)x^{(a-\epsilon_{i'})} + [(1-\delta_{0i})a_i + 1]x^{(a+\epsilon_i-\epsilon_n)}$ *for* $0 \le i < n$.
(2) $\langle h_i, x^{(a)}\rangle = \sigma(i)(a_{i'} - a_i)x^{(a)}$ *for* $1 \le i < n$.
(3) $\langle h_n, x^{(a)}\rangle = \|a\|x^{(a)}$.
(4) *For* $1 \le i,j,k,l < n$, *we have*

$$\begin{aligned}\langle x^{(\epsilon_i+\epsilon_j)}, x^{(\epsilon_k+\epsilon_l)}\rangle = 2^{-\delta_{ij}-\delta_{kl}}\big[&\sigma(j)\big(\delta_{l'j}2^{\delta_{ik}}x^{(\epsilon_i+\epsilon_k)} + \delta_{k'j}2^{\delta_{il}}x^{(\epsilon_i+\epsilon_l)}\big) \\ &+ \sigma(i)\big(\delta_{i'l}2^{\delta_{jk}}x^{(\epsilon_j+\epsilon_k)} + \delta_{i'k}2^{\delta_{jl}}x^{(\epsilon_j+\epsilon_l)}\big)\big].\end{aligned}$$

We have $K = \dot{\sum}_{i\ge -2} K_i$, where $K_i := \langle x^{(a)} \mid \|a\| = i\rangle \cap K$. This is the restricted grading on K referred to in Section 1. The F-space $K_{-2} + K_{-1}$ has basis $\{x_i \mid 0 \le i < n\}$, and an isomorphism from the restricted subalgebra K_0 of K to $\mathfrak{sp}_{2r}(F) \oplus F$ is obtained via

$$\begin{aligned} x^{(\epsilon_i+\epsilon_j)} &\mapsto 2^{-\delta_{ij}}\big(\sigma(i)e_{ji'} + \sigma(j)e_{ij'}\big) \in \mathfrak{sp}_{2r}(F) \quad (1 \le i,j \le 2r), \\ x^{(\epsilon_n)} &\mapsto 1 \in F. \end{aligned}$$

We view $\mathbb{Z}^{2r}$ as a subset of $\mathbb{Z}^n$ by identifying $(a_1,\dots,a_{2r})$ with $(a_1,\dots,a_{2r},0)$ and accordingly we regard $\mathfrak{A}(2r,\mathbf{1})$ as a subspace of $\mathfrak{A}(n,\mathbf{1})$. It is easily checked that $L := \langle x^{(a)} \mid a \in A(2r,\mathbf{1}), a < \tau \in A(2r,\mathbf{1})\rangle$ is a restricted subalgebra of K, and that $J := Fx^{(0)}$ is a restricted ideal of L. The *hamiltonian algebra* is $H = H(2r,\mathbf{1}) = L/J$. We shall write $x^{(a)} + J$ simply as $x^{(a)}$, so that $H = \langle x^{(a)} \mid a \in A(2r,\mathbf{1}), 0 < a < \tau\rangle$. The restricted grading on H of Section 1 is the one induced by that on K, so $H_i = \langle x^{(a)} \mid \|a\| = i\rangle \cap H$. We have $H = \dot{\sum}_{i\ge -1} H_i$. The F-space H_{-1} has basis $\{x_i \mid 1 \le i \le 2r\}$, and the isomorphism $K_0 \to \mathfrak{sp}_{2r}(F) \oplus F$ described above induces an isomorphism $H_0 \to \mathfrak{sp}_{2r}(F)$.

For the remainder of this section, we assume $\mathfrak{g} \in \{K,H\}$. We set $d = n - \delta_{\mathfrak{g}H}$ and

$$\hat{A} = \begin{cases} A(d,\mathbf{1}), & \text{if } \mathfrak{g} = K \text{ and } n+3 \not\equiv 0 \mod p, \\ A(d,\mathbf{1})\backslash\{\tau\}, & \text{otherwise,} \end{cases}$$

so that $\mathfrak{g} = \langle x^{(a)} \mid a \in \hat{A}\rangle$.

In the next lemma and below, we use the notation $T^a := T_1^{a_1}T_2^{a_2}\cdots T_n^{a_n}$ ($a \in \mathbb{Z}^n$, $a \geq 0$), where

$$T_i = \begin{cases} x_i, & 1 \leq i < n, \\ x^{(0)}, & i = n. \end{cases}$$

Also, for $e \in A(2r,\mathbf{1})$, we put $e^1 := (e_1,\dots,e_r,0,\dots,0) \in A(2r,\mathbf{1})$, $e' = (e_{1'},\dots,e_{(2r)'}) \in A(2r,\mathbf{1})$, and $e! = \prod_i(e_i!)$. Finally, if P is a statement, we use the symbol δ_P to represent 1 if the statement is true and 0 if the statement is false.

3.2 Lemma. *Let $\chi \in \mathfrak{g}^*$. The following formulas hold in the algebra $u(\mathfrak{g},\chi)$:*

(1) *For $b \in \hat{A}$ and $0 \leq a \in \mathbb{Z}^n$,*

$$x^{(b)}T^a = \sum_{e,f,k}(-1)^{|e^1+f|+k}2^k\binom{a_n}{k}\binom{a}{e+f}\binom{e+f}{e}\binom{b-(e^1)'+f}{f}f! \cdot T^{a-e-f-k\epsilon_n}x^{(b-e'+f-(|f|+k)\epsilon_n)},$$

where the sum is over all $0 \leq e,f \in \mathbb{Z}^{2r}$ and $0 \leq k \in \mathbb{Z}$ with $e+f+k\epsilon_n \leq a$ and $b-e'+f-(|f|+k)\epsilon_n \in \hat{A}$.

(2) *For $0 \leq a \in \mathbb{Z}^n$ and $1 \leq i < n$,*

$$T^a x_i = T^{a+\epsilon_i} - \delta_{i\leq r}a_{i'}T^{a-\epsilon_{i'}+\epsilon_n}.$$

Proof. (1) We proceed by induction on $|a|$. The case $|a| = 0$ is trivial, so assume $|a| > 0$ and let $1 \leq i \leq n$ be the greatest index for which $a_i \neq 0$.

First suppose $i < n$. Using the induction hypothesis and then 3.1(1) we get

$$\begin{aligned} x^{(b)}T^a &= x^{(b)}T^{a-\epsilon_i}T_i \\ &= \sum_{e,f}(-1)^{|e^1+f|}\binom{a-\epsilon_i}{e+f}\binom{e+f}{e}\binom{b-(e^1)'+f}{f}f!T^{a-e-f}x^{(b-e'+f-|f|\epsilon_n)} \\ &\quad - \sigma(i)\sum_{e,f}(-1)^{|e^1+f|}\binom{a-\epsilon_i}{e+f}\binom{e+f}{e}\binom{b-(e^1)'+f}{f}f!T^{a-\epsilon_i-e-f}x^{(b-e'+f-|f|e_n-\epsilon_{i'})} \\ &\quad - \sum_{e,f}(-1)^{|e^1+f|}\binom{a-\epsilon_i}{e+f}\binom{e+f}{e}\binom{b-(e^1)'+f}{f}f!(b_i-(e')_i+f_i+1) \\ &\qquad \cdot T^{a-\epsilon_i-e-f}x^{(b-e'+f-|f|\epsilon_n+\epsilon_i-\epsilon_n)}, \end{aligned}$$

where the sums are over all $0 \leq e,f \in \mathbb{Z}^{2r}$ with $e+f \leq a-\epsilon_i$ and $b(e,f) := b-e'+f-|f|\epsilon_n \in \hat{A}$ (noting that our assumption on i forces $a_n = 0$ and hence $k = 0$ in each sum). Replacing

e by $e - \epsilon_i$ in the second sum and replacing f by $f - \epsilon_i$ in the third sum, we then get

$$\begin{aligned} x^{(b)}T^a = &\sum_{\substack{e+f \le a-\epsilon_i \\ b(e,f)\in \hat{A}}} \binom{a-\epsilon_i}{e+f}\binom{e+f}{e}\binom{b-(e^1)'+f}{f} f! y(e,f) \\ &+ \sum_{\substack{e\ge \epsilon_i \\ e+f\le a \\ b(e,f)\in \hat{A}-\epsilon_{i'}}} \binom{a-\epsilon_i}{e-\epsilon_i+f}\binom{e-\epsilon_i+f}{e-\epsilon_i}\binom{b-((e-\epsilon_i)^1)'+f}{f} f! y(e,f) \\ &+ \sum_{\substack{f\ge \epsilon_i \\ e+f\le a \\ b(e,f)\in \hat{A}+\epsilon_i-\epsilon_n}} \binom{a-\epsilon_i}{e+f-\epsilon_i}\binom{e+f-\epsilon_i}{e}\binom{b-(e^1)'+f-\epsilon_i}{f-\epsilon_i} \\ &\qquad\qquad \cdot (f-\epsilon_i)!(b_i - e_{i'} + f_i) y(e,f), \end{aligned}$$

where $y(e,f) := (-1)^{|e^1+f|}T^{a-e-f}x^{(b-e'+f-|f|\epsilon_n)}$ and where we have used the fact that $(-1)^{|(e-\epsilon_i)^1|} = -\sigma(i)(-1)^{|e^1|}$.

Let $0 \le e, f \in \mathbb{Z}^{2r}$ with $e+f \le a$. If $i \le r$, then $0 \le f_{i'} \le a_{i'} = 0$ (by the definition of i), and if $i > r$, then $(e-\epsilon_i)^1 = e^1$. Therefore, either $b_{i'} - e_i + f_{i'} < 0$ (in which case $y(e,f) = 0$), or

$$\binom{b-((e-\epsilon_i)^1)'+f}{f} f! = \binom{b-(e^1)'+f}{f} f! =: d(e,f).$$

Now assume $f \ge \epsilon_i$. If $i \le r$, then $0 \le e_{i'} \le a_{i'} = 0$. Hence, $e_{i'} = ((e^1)')_i$ in general, implying

$$\binom{b-(e^1)'+f-\epsilon_i}{f-\epsilon_i}(f-\epsilon_i)!(b_i - e_{i'} + f_i) = d(e,f)$$

as well.

Therefore, observing that we can let e and f range with $e, f \ge 0$, $e+f \le a$, and $b(e,f) \in \hat{A}$ in all three sums (the additional terms contributing nothing), we see that the sums combine to yield

$$x^{(b)}T^a = \sum_{e,f} c(e,f)d(e,f)y(e,f),$$

where

$$c(e,f) = \left[\binom{a-\epsilon_i}{e+f}\binom{e+f}{e} + \binom{a-\epsilon_i}{e-\epsilon_i+f}\binom{e-\epsilon_i+f}{e-\epsilon_i} + \binom{a-\epsilon_i}{e+f-\epsilon_i}\binom{e+f-\epsilon_i}{e}\right].$$

Using a standard binomial coefficient identity twice, we find that $c(e,f) = \binom{a}{e+f}\binom{e+f}{e}$, which completes the proof of the case $i < n$.

It remains to check the case $i = n$. This proof uses the formula $\langle x^{(0)}, x^{(b)} \rangle = 2x^{(b-\epsilon_n)}$, which is 3.1(1) with $i = 0$. The remainder of the proof is routine and is therefore omitted.

(2) Let $1 \leq i < n$. By 3.1(1), we have $T_j T_i = T_i T_j$ for $j \neq i'$, while $T_{i'} T_i = T_i T_{i'} - \sigma(i) T_n$. In particular, the formula is clearly valid if $i > r$. Suppose now $1 \leq i \leq r$. Then for $0 \leq a \in \mathbb{Z}^n$, we have

$$T^a x_i = T^a T_i = \left(\prod_{j \neq i'} T_j^{a_j} \right) (T_{i'}^{a_{i'}} T_i)$$

$$= \left(\prod_{j \neq i'} T_j^{a_j} \right) (T_i T_{i'}^{a_{i'}} - a_{i'} T_{i'}^{a_{i'}-1} T_n) = T^{a+\epsilon_i} - a_{i'} T^{a-\epsilon_{i'}+\epsilon_n},$$

the third equality from the first part of this paragraph and an easy induction on $a_{i'}$. □

The triangular decomposition $\mathfrak{g}_0 = \mathfrak{n}_0^- \dot{+} \mathfrak{h} \dot{+} \mathfrak{n}_0$ of Section 1 is obtained by setting $\mathfrak{n}_0^- = \langle x^{(\epsilon_i+\epsilon_j)} \,|\, (i,j) \in I^- \rangle$, $\mathfrak{h} = \langle h_i \,|\, 1 \leq i \leq d \rangle$, $\mathfrak{n}_0 = \langle x^{(\epsilon_i+\epsilon_j)} \,|\, (i,j) \in I \rangle$, where $I^- = \{(i,j) \,|\, 1 \leq i' < j \leq r \text{ or } r < i \leq j < n\}$ and $I = \{(i,j) \,|\, 1 \leq i \leq j \leq r \text{ or } r < i' < j < n\}$.

For $0 \leq k \leq d$, set

$$\omega_k = -\sum_{i=1}^{\bar{k}} (\epsilon_i + \epsilon_{i'}) + \delta_{\mathfrak{g}K} \left[\sigma(k)(r + 1 - \bar{k}) - r - 1 \right] \epsilon_n \in \Lambda,$$

where

$$\bar{k} = \begin{cases} k, & 0 \leq k \leq r, \\ k', & r < k \leq n. \end{cases}$$

Then $\Lambda_e = \{\omega_0, \omega_1, \ldots, \omega_d\}$ is the set of exceptional weights.

We point out that in this paper weights are d-tuples, whereas in [H1–4], they were $(r + \delta_{\mathfrak{g}K})$-tuples. Since $h_{i'} = x^{(\epsilon_{i'}+\epsilon_i)} = h_i$ $(1 \leq i \leq r)$, a d-tuple $\lambda \in F^d$ is a weight of a nonzero vector only if $\lambda_{i'} = \lambda_i$ for each $1 \leq i \leq r$. Therefore, there is redundancy here and r entries of λ can be dropped to give a weight in the earlier sense. We use the d-tuples because they provide a certain flexibility useful for simplifying arguments. If $\mathfrak{g} = H$, then $\omega_{k'} = \omega_k$ for $1 \leq k \leq r$, implying $\Lambda_e = \{\omega_0, \omega_1, \ldots, \omega_r\}$ (cf. [H4]). If $\mathfrak{g} = K$, then ω_k corresponds to ω_k^+ for $0 \leq k \leq r$ and to $\omega_{k'}^-$ for $r < k \leq n$, where $\omega_k^\pm$ are as in [H3].

To begin the proof of 1.3, we assume its hypothesis: Let $\chi \in \mathfrak{g}^*$ with $\operatorname{ht} \chi \leq 1$, let M be a $u(\mathfrak{g}_0, \chi)$-module, let $v \in Z^\chi(M)$ be a maximal vector of weight $\lambda \in \Lambda^\chi$ and assume that either $\chi(\mathfrak{n}_0^-) \neq 0$ or M has no maximal vector of exceptional weight.

Putting $A = A(d, \mathbf{1})$ we have, just as in Section 2, that $v = \sum_{a \in A} T^a \otimes m_a$ with the m_a uniquely determined elements of M.

For $a \in A$, define $\lambda(a) \in F^d$ by setting, for $1 \leq i \leq d$,

$$\lambda(a)_i = \begin{cases} \lambda_i + \sigma(i)(a_i - a_{i'}), & \text{if } 1 \leq i < n, \\ \lambda_n + |a| + a_n, & \text{if } i = n. \end{cases}$$

3.3 Lemma. *If $a \in A$, then $h_i \cdot m_a = \lambda(a)_i m_a$ for $1 \le i \le d$.*

Proof. Let $1 \le i \le d$. First assume $i \le r$. Using 3.2(1), we get

$$\begin{aligned} h_i \cdot v = x^{(\epsilon_i+\epsilon_{i'})} \cdot v &= \sum_a x^{(\epsilon_i+\epsilon_{i'})} T^a \otimes m_a \\ &= \sum_a T^a x^{(\epsilon_i+\epsilon_{i'})} \otimes m_a - \sum_a a_i T^{a-\epsilon_i} x_i \otimes m_a \\ &\quad + \sum_a a_{i'} T^{a-\epsilon_{i'}} x_{i'} \otimes m_a - \sum_a a_i a_{i'} T^{a-\epsilon_i-\epsilon_{i'}} x^{(0)} \otimes m_a. \end{aligned}$$

The final four sums come from the terms in 3.2(1) corresponding to $f = 0$, $k = 0$, and $e = 0, \epsilon_i, \epsilon_{i'}, \epsilon_i + \epsilon_{i'}$, respectively, which are the only choices of (e, f, k) that satisfy the constraints. Applying 3.2(2) to the second and third sums and then collecting terms gives

$$h_i \cdot v = \sum_a T^a \otimes (h_i - a_i + a_{i'}) \cdot m_a.$$

The desired formula now follows just as in the proof of 2.2. We then obviously have the formula for $r < i < n$ as well. If $i = n$ (so that $\mathfrak{g} = K$), then a similar proof gives the formula. (Here, we need to use 3.2(1) to rewrite $x^{(\epsilon_n)} T^a$ and the only indices we need to include are $(e, f, k) = (0, 0, 0), (0, 0, 1), (0, \epsilon_i, 0), (\epsilon_{i'}, \epsilon_i, 0)$, for $1 \le i < n$.) $\square$

Just as in the discussion before 2.3, it is convenient to extend the definition of m_a: If $b \in \mathbb{Z}^n$, and $b \not\le \tau \in A$, put $m_b = 0$; if $a \in A$ and $a_i = 0$, put $m_{a-\epsilon_i} = \chi(T_i)^p m_{a+(p-1)\epsilon_i}$. Observe that if $\mathfrak{g} = H$, and $a \in \mathbb{Z}^d \subset \mathbb{Z}^n$, then $m_{a-\epsilon_n} = 0$ since $T_n = x^{(0)} = 0$. In particular, the final term in each of the formulas 3.4(1,2,4,5,6) below vanishes if $\mathfrak{g} = H$.

3.4 Lemma. *Let $1 \le i, j < n$ and let $b \in A$.*

(1) *If $3 \le l < p$, or if $2 \le l < p$ and $i \le r$, then*

$$\begin{aligned} 0 = &\binom{b_{i'} + l - 2}{l - 2} x^{(2\epsilon_i)} \cdot m_{b+(l-2)\epsilon_{i'}} + \sigma(i) \binom{b_{i'} + l - 1}{l - 1} m_{b+(l-1)\epsilon_{i'}-\epsilon_i} \\ &+ \binom{b_{i'} + l}{l} (1 - l\delta_{i\le r}) m_{b+l\epsilon_{i'}-\epsilon_n}. \end{aligned}$$

(2) *If $1 \le i, j \le r$, then*

$$\begin{aligned} 0 = &2^{\delta_{ij}} x^{(\epsilon_i+\epsilon_j)} \cdot m_b + (b_{i'} + 1) m_{b-\epsilon_j+\epsilon_{i'}} \\ &+ (b_{j'} + 1) m_{b-\epsilon_i+\epsilon_{j'}} - (b_{i'} + \delta_{ij} + 1)(b_{j'} + 1) m_{b+\epsilon_{i'}+\epsilon_{j'}-\epsilon_n}. \end{aligned}$$

(3) *If $r < i' < j < n$, then*

$$0 = x^{(\epsilon_i+\epsilon_j)} \cdot m_b - (b_{j'} + 1) m_{b-\epsilon_i+\epsilon_{j'}} + (b_{i'} + 1) m_{b+\epsilon_{i'}-\epsilon_j}.$$

(4) *If $j \notin \{i, i'\}$, then*

$$0 = \sigma(i)(b_{i'} + 1)x^{(\epsilon_i+\epsilon_j)} \cdot m_{b+\epsilon_{i'}} + \sigma(j)(b_{j'} + 1)x^{(2\epsilon_i)} \cdot m_{b+\epsilon_{j'}} + \binom{b_{i'}+2}{2} m_{b+2\epsilon_{i'}-\epsilon_j} + \sigma(i)\sigma(j)(b_{i'} + 1)(b_{j'} + 1)m_{b+\epsilon_{i'}+\epsilon_{j'}-\epsilon_i} - (\delta_{j>r} + 2\sigma(j)\delta_{i\le r})\binom{b_{i'}+2}{2}(b_{j'} + 1)m_{b+2\epsilon_{i'}+\epsilon_{j'}-\epsilon_n}.$$

(5) *We have*

$$0 = \left[\binom{b_{i'}+1}{2} - b_i(b_{i'} + 1) + \sigma(i)(b_{i'} + 1)\lambda(b + \epsilon_{i'})_i\right] m_{b+\epsilon_{i'}} - \sigma(i)(b_i + 1)x^{(2\epsilon_i)} \cdot m_{b+\epsilon_i} + \delta_{i\le r}(b_i + 1)\binom{b_{i'}+2}{2} m_{b+\epsilon_i+2\epsilon_{i'}-\epsilon_n}.$$

(6) *If $1 \le i \le r$ and $j \notin \{i, i'\}$, then*

$$0 = (b_i + 1)(b_{i'} + 1)m_{b+\epsilon_i+\epsilon_{i'}-\epsilon_j} - (b_{i'} + 1)x^{(\epsilon_{i'}+\epsilon_j)} \cdot m_{b+\epsilon_{i'}} + (b_i + 1)x^{(\epsilon_i+\epsilon_j)} \cdot m_{b+\epsilon_i} - \sigma(j)(b_{j'} + 1)[\lambda(b + \epsilon_{j'})_i - b_i + b_{i'}]m_{b+\epsilon_{j'}} - \delta_{j\le r}(b_i + 1)(b_{i'} + 1)b_{j'} m_{b+\epsilon_i+\epsilon_{i'}+\epsilon_{j'}-\epsilon_n}.$$

(7) *If $\mathfrak{g} = K$, then*

$$0 = (b_{i'} + 1)\big[b_n - b_i - \lambda(b + \epsilon_{i'})_i + \sigma(i)\lambda(b + \epsilon_{i'})_n - \sigma(i)|b|\big]m_{b+\epsilon_{i'}} - \sum_{\substack{1\le j<n \\ j\ne i'}} (1 + \delta_{ij})(b_j + 1)x^{(\epsilon_i+\epsilon_j)} \cdot m_{b+\epsilon_j} - 2(b_n + 1)m_{b-\epsilon_i+\epsilon_n} - \sum_{\substack{1\le j\le r \\ j\notin\{i,i'\}}} (b_{j'} + 1)(b_j + 1)m_{b+\epsilon_{j'}+\epsilon_j-\epsilon_i} + \delta_{i\le r}\sum_{j=1}^{r}(b_{i'} + \delta_{ji'} + \delta_{ji} + 1)(b_j + \delta_{ji'} + 1)(b_{j'} + 1)m_{b+\epsilon_i+\epsilon_j+\epsilon_{j'}-\epsilon_n}.$$

Proof. (1) Assuming either of the stated conditions, we have $x^{(l\epsilon_i)} \in \mathfrak{n}$. Therefore, using that $\mathfrak{n} \cdot v = 0$ and then 3.2(1), we obtain

$$0 = x^{(l\epsilon_i)} \cdot v = \sum_a \sigma(i)^{l-2}\binom{a_{i'}}{l-2} T^{a-(l-2)\epsilon_{i'}} x^{(2\epsilon_i)} \otimes m_a + \sum_a \sigma(i)^{l-1}\binom{a_{i'}}{l-1} T^{a-(l-1)\epsilon_{i'}} x^{(\epsilon_i)} \otimes m_a + \sum_a \sigma(i)^{l}\binom{a_{i'}}{l} T^{a-l\epsilon_{i'}} x^{(0)} \otimes m_a.$$

The sums come from the terms in 3.2(1) corresponding to $f = 0$, $k = 0$, and $e = (l-2)\epsilon_{i'}, (l-1)\epsilon_{i'}, l\epsilon_{i'}$, respectively; all other choices of (e, f, k) make no contribution, either because the exponent of $x^{(l\epsilon_i - e' + f - (|f|+k)\epsilon_n)}$ is not in $\hat{A}$, or because this element is in $\mathfrak{g}^1$ which annihilates M. Using 3.2(2) on the second sum and combining, we obtain

$$0 = \sum_a T^{a-(l-2)\epsilon_{i'}} \otimes \binom{a_{i'}}{l-2} x^{(2\epsilon_i)} \cdot m_a + \sigma(i) \sum_a T^{a-(l-1)\epsilon_{i'}+\epsilon_i} \otimes \binom{a_{i'}}{l-1} m_a$$
$$+ \sum_a T^{a-l\epsilon_{i'}+\epsilon_n} \otimes [1 - l\delta_{i\le r}] \binom{a_{i'}}{l} m_a.$$

Now a shift of indices expresses each sum in the form $\sum_b T^b \otimes n_b$ with $n_b \in M$, and the conventions given before the statement of this lemma allow the sum to be taken over all $b \in A$ (see proof of 2.3). Combining the sums and using uniqueness of expression, we get the desired formula.

The remaining formulas are proved in a similar fashion. We shall just indicate how to begin the proofs and leave the details to the interested reader.

(2 and 3) Use that $x^{(\epsilon_i+\epsilon_j)} \cdot v = 0$ when $(i, j) \in I$ since v is a maximal vector.

(4,5, and 6) Use that $x^{(2\epsilon_i+\epsilon_j)} \cdot v = 0$, $x^{(2\epsilon_i+\epsilon_{i'})} \cdot v = 0$, $x^{(\epsilon_i+\epsilon_{i'}+\epsilon_j)} \cdot v = 0$, respectively, since $\mathfrak{g}^1 \cdot M = 0$.

(7) Assuming $\mathfrak{g} = K$, we have $x^{(\epsilon_i+\epsilon_n)} \in \mathfrak{g}^1$, so $x^{(\epsilon_i+\epsilon_n)} \cdot v = 0$. □

We are now ready to carry out the main portion of the proof of 1.3, which amounts to showing that in our expression $v = \sum_a T^a \otimes m_a$, each m_a is zero unless $a = 0$. We proceed in steps.

3.5 Lemma. *Let $a \in A$ and let $1 \le i < n$. If $a_i \ne p-1$ and $a_{i'} \ge 4$, then $m_a = 0$.*

Proof. Assume the hypothesis. First suppose $p > 5$. Our assumptions imply $b := a - (l-1)\epsilon_{i'} + \epsilon_i \in A$ for $l \in \{3, 4, 5\}$. Therefore, 3.4(1) applies to yield a system of three equations (one for each l),

$$0 = \binom{a_{i'}-1}{l-2} x^{(2\epsilon_i)} \cdot m_{a-\epsilon_{i'}+\epsilon_i} + \sigma(i) \binom{a_{i'}}{l-1} m_a$$
$$+ \binom{a_{i'}+1}{l} (1 - l\delta_{i\le r}) m_{a+\epsilon_{i'}+\epsilon_i-\epsilon_n} \tag{3.5.1}$$

which one routinely solves to find that $m_a = 0$.

Now suppose $p = 5$. Then $4 \le a_{i'} \le p-1$ forces $a_{i'} = 4$. This implies $m_{a+\epsilon_i+\epsilon_{i'}-\epsilon_n} = 0$, so the first two equations in the above system, which are still valid for this p, involve only $m_{a-\epsilon_{i'}+\epsilon_i}$ and m_a. The resulting system again yields $m_a = 0$. □

3.6 Lemma. *Let $a \in A$ and let $1 \leq i < n$. If $a_{i'} \geq 3$ and either $a_{i'} \neq p-1$ or $a_i \neq p-1$, then $m_a = 0$.*

Proof. Assume the hypothesis. If $a_i = p-1$, then $a_{i'} \neq p-1$ and also $a_i \geq 4$ (since $p \geq 5$), so 3.5 applies to give $m_a = 0$. Now assume $a_i \neq p-1$. By 3.5, we may assume $a_{i'} = 3$. Suppose $a_i = p-2$. Then $a_i \geq 3$ (again, since $p \geq 5$), implying $m_{a-\epsilon_{i'}+\epsilon_i} = 0$ by 3.5. Under our assumptions, the first two equations in (3.5.1) remain valid and comprise a system in m_a and $m_{a+\epsilon_i+\epsilon_{i'}-\epsilon_n}$. This system yields $m_a = 0$. Finally, suppose $a_i \neq p-2$. Then 3.5 implies $m_{a+\epsilon_i+\epsilon_{i'}-\epsilon_n} = 0$, so the first two equations in (3.5.1) comprise a system in m_a and $m_{a-\epsilon_{i'}+\epsilon_i}$, which again yields $m_a = 0$. □

3.7 Lemma. *If $a \in A$ and $a_i = p-1$ for some $1 \leq i < n$, then $m_a = 0$.*

Proof. We prove the following claim: If $a \in A$ and $a_i = p-1 = a_{i'}$, $a_j \neq p-1$ for some $1 \leq i \leq r$, $1 \leq j < n$, with $j \notin \{i, i'\}$, then $m_a = 0$.

Assume the hypothesis and let $b = a - \epsilon_i - \epsilon_{i'} + \epsilon_j$ in 3.4(6). The first term on the right becomes m_a, by 3.5 and 3.6 the next three terms are zero, and the last term is a multiple of m_c where $c = a + \epsilon_j + \epsilon_{j'} - \epsilon_n$ if $a_n \neq 0$ and $c = a + \epsilon_j + \epsilon_{j'} + (p-1)\epsilon_n$ if $a_n = 0$.

We prove our claim by reverse induction on a_j. If $a_j = p-2$, then $m_a = 0$ by 3.6. Now suppose $a_j < p-2$. If $a_{j'} = p-1$, then $m_a = 0$ by 3.5, so suppose $a_{j'} < p-1$. Then $c \in A$, $c_i = p-1 = c_{i'}$, $c_j \neq p-1$, and $c_j = a_j + 1 > a_j$. Therefore, $m_c = 0$ by the induction hypothesis. Hence $m_a = 0$ and the claim is established.

Now suppose there exists $a \in A$ such that $a_i = p-1$ for some $1 \leq i < n$, and $m_a \neq 0$. We shall derive a contradiction. By 3.5, $a_{i'} = p-1$. Then the first part of the proof applies to give $a_i = p-1$ for all $1 \leq i < n$.

Next, we argue that m_a is a maximal vector with the exceptional weight ω_n.

Letting $b = a$ in both 3.4(2) and 3.4(3) we get $x^{(\epsilon_i+\epsilon_j)} \cdot m_a = 0$ for all $(i,j) \in I$ (since each of the other terms is a multiple of m_c with $c \not\leq \tau$), so m_a is a maximal vector (recalling that $\mathfrak{g}^1 \cdot M = 0$).

Letting $b = a - \epsilon_{i'}$ $(1 \leq i < n)$ in 3.4(5), we see that the last two terms are zero, which forces the coefficient of $m_{b+\epsilon_{i'}}$ $(= m_a)$ to be zero. Thus $\lambda(a)_i = 0$ for each $0 \leq i < n$. And if $\mathfrak{g} = K$, then putting $b = a - \epsilon_{i'}$ (any $1 \leq i \leq r$) in 3.4(7), we see that the terms past the first are zero (noting that the ith component of $b - \epsilon_i + \epsilon_n$ is $p-2$, implying $m_{b-\epsilon_i+\epsilon_n} = 0$ by 3.6), so the coefficient of $m_{b+\epsilon_{i'}}$ $(= m_a)$ is zero, whence $\lambda(a)_n = -2r-2$. We conclude, using 3.3, that m_a is a maximal vector of exceptional weight ω_n.

Checking the hypotheses of the theorem (1.3), we see that it must be the case that $\chi(\mathfrak{n}_0^-) \neq 0$. Hence, $\chi(x^{(\epsilon_i+\epsilon_j)}) \neq 0$ for some $(i,j) \in I^-$. Now putting $b = a - \epsilon_{i'}$ in 3.4(4), or in 3.4(1) with $l = 3$ in the case $i = j$, we get $x^{(\epsilon_i+\epsilon_j)} \cdot m_a = 0$. Then $\chi(x^{(\epsilon_i+\epsilon_j)})^p m_a = (x^{(\epsilon_i+\epsilon_j)})^p \cdot m_a = 0$, which is a contradiction since $\chi(x^{(\epsilon_i+\epsilon_j)}) \neq 0$ and $m_a \neq 0$. This contradiction establishes the lemma. □

3.8 Lemma. *Let $a \in A$. If $a_i \geq 3$ for some $1 \leq i < n$ or $a_{i'} \geq 2$ for some $1 \leq i \leq r$, then $m_a = 0$.*

Proof. Assume $a_i \geq 3$ for some $1 \leq i < n$. If $a_{i'} = p-1$, then 3.7 says $m_a = 0$, while if $a_{i'} \neq p-1$, then 3.6 says $m_a = 0$. It remains to consider the case $a_{i'} = 2$ for some $1 \leq i \leq r$, which we now assume. By 3.7, we may assume $a_i \neq p-1$. Since $m_{a+\epsilon_i+\epsilon_{i'}-\epsilon_n} = 0$ by the first part of this proof, we obtain, by letting $b = a - (l-1)\epsilon_{i'} + \epsilon_i$ and $l \in \{2, 3\}$ in 3.4(1), a system in m_a and $m_{a-\epsilon_{i'}+\epsilon_i}$ (see (3.5.1)) that implies $m_a = 0$. □

3.9 Lemma. *Let $a \in A$. If $a_i \neq 0$, $a_{i'} \neq 0$, and $a_j \neq 0$ for some $1 \leq i \leq r$, $1 \leq j < n$, with $j \notin \{i, i'\}$, then $m_a = 0$.*

Proof. First suppose $a_i \geq 2$ and $a_{i'} \neq 0$ for some $1 \leq i \leq r$. If $a_i = p-1$, then $m_a = 0$ by 3.7. And if $a_i \neq p-1$, then 3.4(1) with $b = a - \epsilon_{i'} + \epsilon_i$ and $l = 2$ shows, in light of 3.8, that $m_a = 0$.

Now assume the hypotheses of this lemma. By 3.7 we may assume $a_i \neq p-1$. First assume $j > r$. Replacing j by j' in 3.4(2) and then putting $b = a + \epsilon_i - \epsilon_j$ we find that the third term becomes $a_j m_a$, and the other terms are zero by the first observation of this proof. Therefore, $m_a = 0$. Now assume $j \leq r$. If $i < j$, replace j by j' in 3.4(3) and then put $b = a + \epsilon_i - \epsilon_j$ to get $m_a = 0$ (again using the first observation of this proof). So now suppose $j < i$. By 3.7, we may assume $a_{j'} \neq p-1$. In 3.4(4), replace i by j', replace j by i, and put $b = a - \epsilon_j - \epsilon_{i'} + \epsilon_{j'}$. Then the fourth term becomes $-a_j a_{i'} m_a$ and the other terms are zero. Indeed, $m_{b+\epsilon_j} = m_{a-\epsilon_{i'}+\epsilon_{j'}} = 0$ by the case $i < j$ just established, $m_{b+\epsilon_{i'}} = m_{a-\epsilon_j+\epsilon_{j'}} = 0$ by the case $j > r$ established above, and the other two terms are zero by the first observation of this proof. Thus, $m_a = 0$ and the proof is complete. □

3.10 Lemma. *Let $a \in A$. If $a_i = 2$ for some $1 \leq i \leq r$ and $a \neq 2\epsilon_i$, then $m_a = 0$.*

Proof. Assume $a_i = 2$ for some $1 \leq i \leq r$. Suppose $a_j \neq 0$ for some $1 \leq j < n$ with $j \neq i$. If $j = i'$, then $m_a = 0$ by the first part of the proof of 3.9. Now assume $j \notin \{i, i'\}$. By 3.7 we may assume $a_{i'} \neq p-1$. Replace i and j in 3.4(4) by i' and j', respectively, and then put $b = a - \epsilon_i - \epsilon_j + \epsilon_{i'}$. The fourth term becomes $\sigma(j) a_i a_j m_a$ and the other terms are zero ($m_{b+\epsilon_i} = m_{a-\epsilon_j+\epsilon_{i'}} = 0$ by the first observation of this proof, $m_{b+\epsilon_j} = m_{a-\epsilon_i+\epsilon_{i'}} = 0$ by 3.9, the third and fifth terms are zero by 3.8). Therefore, $m_a = 0$. Finally, suppose $a_n \neq 0$. Then putting $b = a + \epsilon_i - \epsilon_n$ in 3.4(7), we find that the third term becomes $-2a_n m_a$ and the other terms are zero (using 3.8 for the first and second terms and 3.9 for the others). Therefore, we have shown that if $a \neq 2\epsilon_i$, then $m_a = 0$, as desired. □

3.11 Lemma. *Let $a \in A$. If $a_i \geq 2$ for some $1 \leq i < n$, then $m_a = 0$.*

Proof. We begin by making some general observations. To simplify notation, we set $m_k := m_{\epsilon_k+\epsilon_{k'}}$ $(1 \leq k \leq r)$. Putting $b = \epsilon_k + \epsilon_{k'}$ in 3.4(2 and 3) and using 3.7, 3.8, and 3.9, we obtain

(3.11.1) $$x^{(\epsilon_i+\epsilon_j)} \cdot m_k = \big(\sigma(j')\delta_{ik} - \delta_{jk} - \delta_{j'k}\big)m_{\epsilon_{i'}+\epsilon_{j'}}$$

whenever $1 \le k \le r$ and $(i,j) \in I$. Next, with j replaced by j' in 3.4(4) and with $b = \epsilon_i$, we get (using 3.8)

$$0 = \sigma(j')x^{(\epsilon_i+\epsilon_{j'})} \cdot m_i + x^{(2\epsilon_i)} \cdot m_{\epsilon_i+\epsilon_j} + m_{\epsilon_j+\epsilon_{i'}}$$

for $1 \le i \le r$, $j \notin \{i, i'\}$, while putting $l = 2$, $b = \epsilon_i + \epsilon_j$ in 3.4(1) we get (using 3.8)

$$0 = x^{(2\epsilon_i)} \cdot m_{\epsilon_i+\epsilon_j} + m_{\epsilon_j+\epsilon_{i'}}$$

for $1 \le i \le r$, $j \notin \{i, i'\}$. Therefore, if $(i,j) \in I$ with $i \ne j$, then

(3.11.2) $$m_{\epsilon_{i'}+\epsilon_{j'}} = \sigma(j')x^{(\epsilon_i+\epsilon_j)} \cdot m_i = 0,$$

where the first equality is from (3.11.1) with $k = i$ and the second is from a combination of the last two formulas. We then conclude from (3.11.1) (and 3.8 for the case $i = j$ to see that $m_{2\epsilon_{i'}} = 0$) that

(3.11.3) $$x^{(\epsilon_i+\epsilon_j)} \cdot m_k = 0$$

whenever $1 \le k \le r$ and $(i,j) \in I$. In other words, m_k is a maximal vector if it is nonzero.

Next, we derive some formulas that will be used to determine the weight of m_k. Let $1 \le i \le r$. For any $1 \le j \le r$, we set $\lambda_{ji} := \lambda(\epsilon_j + \epsilon_{j'})_i$. First, $b = \epsilon_i$ in 3.4(5) yields (using 3.8)

$$0 = -2x^{(2\epsilon_i)} \cdot m_{2\epsilon_i} + (\lambda_{ii} - 1)m_i,$$

while $l = 2$, $b = 2\epsilon_i$ in 3.4(1) yields (using 3.8 again)

(3.11.4) $$0 = x^{(2\epsilon_i)} \cdot m_{2\epsilon_i} + m_i.$$

Combining these two equations, we obtain

(3.11.5) $$\lambda_{ii}m_i = -m_i.$$

Now let $1 \le i, j \le r$, $j \ne i$. From 3.4(6) with $b = \epsilon_j$ we get (using 3.9 to see that the last term is zero)

$$0 = m_i - x^{(\epsilon_{i'}+\epsilon_j)} \cdot m_{\epsilon_{i'}+\epsilon_j} + x^{(\epsilon_i+\epsilon_j)} \cdot m_{\epsilon_i+\epsilon_j} - \lambda_{ji}m_j,$$

while from 3.4(2) with $b = \epsilon_i + \epsilon_j$ we get (using 3.9 again)

$$0 = x^{(\epsilon_i+\epsilon_j)} \cdot m_{\epsilon_i+\epsilon_j} + m_i + m_j, \tag{3.11.6}$$

so

$$0 = x^{(\epsilon_{i'}+\epsilon_j)} \cdot m_{\epsilon_{i'}+\epsilon_j} + (\lambda_{ji} + 1)m_j. \tag{3.11.7}$$

But $b = \epsilon_i + \epsilon_{j'}$ in 3.4(3) with j replaced by j' yields

$$0 = x^{(\epsilon_i+\epsilon_{j'})} \cdot m_{\epsilon_i+\epsilon_{j'}} + m_i - m_j \tag{3.11.8}$$

whenever $i < j$. Hence

$$\lambda_{ji} m_j = \begin{cases} -m_j, & 1 \le i \le j \le r, \\ -m_i, & 1 \le j \le i \le r, \end{cases} \tag{3.11.9}$$

the first case from (3.11.7), (3.11.2), (3.11.5) and the second case from (3.11.7), (3.11.8), (3.11.5).

Now assume the hypothesis of this lemma: Assume $a_k \ge 2$ for some $1 \le k < n$. By 3.8 and 3.10, we may assume $1 \le k \le r$ and $a = 2\epsilon_k$. Suppose $m_a \ne 0$. Replacing i by i' in 3.4(1) and then putting $l = 3$, $b = \epsilon_{i'}$, we find (using 3.8) that

$$0 = x^{(2\epsilon_{i'})} \cdot m_i - m_{2\epsilon_i}, \tag{3.11.10}$$

for $1 \le i \le r$, whence $m_k \ne 0$. By replacing k if necessary we may assume that k is the greatest index ($1 \le k \le r$) for which $m_k \ne 0$ (since, in view of (3.11.4), we will still have $m_a \ne 0$). By (3.11.9),

$$\lambda_{ki} = \begin{cases} -1, & 1 \le i \le k, \\ 0, & k < i \le r, \end{cases} \tag{3.11.11}$$

so, in particular, if $\mathfrak{g} = H$, then m_k has the exceptional weight ω_k.

Now suppose $\mathfrak{g} = K$. We shall first compute λ_{kn} and then use the result to show that m_k has exceptional weight in this case as well.

Let $1 \le j < k$. By (3.11.9), we have $\lambda_{jk} m_j = -m_k$, which implies $m_j \ne 0$. Now (3.11.11) says $\lambda_{kk} = -1$, so $\lambda_{jk} = \lambda_k = \lambda_{kk} = -1$ (see definition of $\lambda(a)_i$ before 3.3), whence $m_j = m_k$. We conclude that

$$m_j = \begin{cases} m_k, & 1 \le j \le k, \\ 0, & k < j \le r. \end{cases} \tag{3.11.12}$$

Next, replacing i by k' in 3.4(7) and putting $b = \epsilon_k$ we get (using 3.7)

$$
\begin{aligned}
0 = -\sum_{\substack{1\le j<n\\ j\notin\{k,k'\}}} x^{(\epsilon_{k'}+\epsilon_j)}\cdot m_{\epsilon_k+\epsilon_j} - 2x^{(2\epsilon_{k'})}\cdot m_k \\
+ 2\,[1-\lambda(2\epsilon_k)_{k'} - \lambda(2\epsilon_k)_n]\, m_{2\epsilon_k}.
\end{aligned} \tag{3.11.13}
$$

Then, replacing i by k' in 3.4(4) and putting $b = \epsilon_j$ we get (using 3.7 and 3.9)

$$x^{(\epsilon_{k'}+\epsilon_j)}\cdot m_{\epsilon_k+\epsilon_j} = \sigma(j)x^{(2\epsilon_{k'})}\cdot m_j + m_{2\epsilon_k}$$

for $1 \le j < n$, $j \notin \{k, k'\}$. In view of (3.11.10) and (3.11.12), this equation becomes

$$x^{(\epsilon_{k'}+\epsilon_j)}\cdot m_{\epsilon_k+\epsilon_j} = \begin{cases} (\sigma(j)+1)m_{2\epsilon_k}, & 1\le j<k \text{ or } r<j<k', \\ m_{2\epsilon_k}, & k<j\le r \text{ or } k'<j<n. \end{cases}$$

Now, from the definition before 3.3 and (3.11.11), we get

$$\lambda(2\epsilon_k)_{k'} = \lambda_k + 2 = \lambda_{kk} + 2 = 1. \tag{3.11.14}$$

Putting these last results in (3.11.13) and using (3.11.10) again produces $\lambda(2\epsilon_k)_n m_{2\epsilon_k} = -rm_{2\epsilon_k}$. Hence,

$$\lambda_{kn} = \lambda_n + 2 = \lambda(2\epsilon_k)_n = -r. \tag{3.11.15}$$

Continuing, $b = \epsilon_i$ in 3.4(7) gives (using 3.9 and 3.10)

$$0 = \sum_{\substack{1\le j<n\\ j\ne i'}} (-1-\delta_{ij})(1+\delta_{ij})x^{(\epsilon_i+\epsilon_j)}\cdot m_{\epsilon_i+\epsilon_j} - \sum_{\substack{1\le j\le r\\ j\ne i}} m_j + [\lambda_{in} - \lambda_{ii} - 2]m_i - 2m_{\epsilon_n}$$

for each $1 \le i \le r$. But (3.11.2), (3.11.6), (3.11.8) combine to give

$$x^{(\epsilon_i+\epsilon_j)}\cdot m_{\epsilon_i+\epsilon_j} = \begin{cases} -m_i - \sigma(j)m_j, & \text{if } 1\le i,j\le r,\ j\ne i \text{ or } 1\le i<j'\le r, \\ 0, & \text{if } r<i,j<n,\ j\ne i \text{ or } 1\le j'<i\le r. \end{cases}$$

Applying this, as well as (3.11.4), to the previous equation and rearranging, we obtain

$$\sum_{i<j\le r} m_j = (2r - i + 1 - \lambda_{ii} + \lambda_{in})m_i - 2m_{\epsilon_n} \tag{3.11.16}$$

for $1 \le i \le r$.

Recall that k is the greatest index with $1 \le k \le r$ for which $m_k \ne 0$. If $k = r$, then m_k is a maximal vector (by (3.11.3)) with the exceptional weight ω_r by (3.11.11) and (3.11.15). Now suppose $k < r$. Putting $i = r$ in (3.11.16) yields $m_{\epsilon_n} = 0$ (using maximality of k).

Then putting $i = k$ in (3.11.16) yields, in view of (3.11.11) and (3.11.15), $(r-k+2)m_k = 0$, so that $r - k + 2 = 0$ (in F). Hence,

$$\lambda_{kn} = -r = -(r + 1 - k) - r - 1$$

and so m_k is a maximal vector with the exceptional weight $\omega_{k'}$ (using (3.11.11) again).

Once again let $\mathfrak{g} \in \{K, H\}$ be arbitrary. We have shown that M has a maximal vector with exceptional weight. Checking the assumptions of 1.3 we see that it must be the case that $\chi(\mathfrak{n}_0^-) \neq 0$, that is, $\chi(x^{(\epsilon_i+\epsilon_j)}) \neq 0$ for some $(i, j) \in I^-$. We claim that $m_{2\epsilon_{i'}} = 0$.

If $j = i$, then $x^{(\epsilon_i+\epsilon_j)} \cdot m_{2\epsilon_{i'}} = x^{(2\epsilon_i)} \cdot m_{2\epsilon_{i'}} = 0$, the last equality from 3.4(1) with $l = 3$, $b = 0$ (using 3.8). This implies $m_{2\epsilon_{i'}} = 0$ when $j = i$.

Now assume $j \neq i$. Let $1 \leq s, t < n$, $t \notin \{s, s'\}$ and replace i and j in 3.4(4) by s and t, respectively. Putting $b = \epsilon_{s'}$ yields

$$0 = 2\sigma(s)x^{(\epsilon_s+\epsilon_t)} \cdot m_{2\epsilon_{s'}} + \sigma(t)x^{(2\epsilon_s)} \cdot m_{\epsilon_{s'}+\epsilon_{t'}},$$

while putting $b = \epsilon_{t'}$ yields

$$0 = \sigma(s)x^{(\epsilon_s+\epsilon_t)} \cdot m_{\epsilon_{s'}+\epsilon_{t'}} + 2\sigma(t)x^{(2\epsilon_s)} \cdot m_{2\epsilon_{t'}}$$

(using 3.10 and 3.8). Since $i' \leq r$, we obtain from the above equations and 3.1(4)

$$\begin{aligned} x^{(\epsilon_i+\epsilon_j)}x^{(\epsilon_i+\epsilon_j)} \cdot m_{2\epsilon_{i'}} &= 2^{-1}\sigma(j)x^{(\epsilon_i+\epsilon_j)}x^{(2\epsilon_i)} \cdot m_{\epsilon_{i'}+\epsilon_{j'}} = 2^{-1}\sigma(j)x^{(2\epsilon_i)}x^{(\epsilon_i+\epsilon_j)} \cdot m_{\epsilon_{i'}+\epsilon_{j'}} \\ &= x^{(2\epsilon_i)}x^{(2\epsilon_j)} \cdot m_{2\epsilon_{i'}} = x^{(2\epsilon_j)}x^{(2\epsilon_i)} \cdot m_{2\epsilon_{i'}} = 0, \end{aligned}$$

the last equality since $x^{(2\epsilon_i)} \cdot m_{2\epsilon_{i'}} = 0$ (again by 3.4(1)). This implies $m_{2\epsilon_{i'}} = 0$.

Now (3.11.4) implies $m_{i'} = 0$, so that $i' > k$ by (3.11.12). According to 3.1(4), we have $x^{(\epsilon_j+\epsilon_i)} = \langle x^{(\epsilon_k+\epsilon_j)}, x^{(\epsilon_{k'}+\epsilon_i)}\rangle$ and, since $x^{(\epsilon_k+\epsilon_j)} \cdot m_k = 0$ (by (3.11.3)), it follows that

$$x^{(\epsilon_i+\epsilon_j)} \cdot m_k = x^{(\epsilon_k+\epsilon_j)}x^{(\epsilon_{k'}+\epsilon_i)} \cdot m_k. \tag{3.11.17}$$

By 3.4(4) with i replaced by k', j replaced by i, and with $b = \epsilon_{k'}$, we have

$$x^{(\epsilon_{k'}+\epsilon_i)} \cdot m_k = -x^{(2\epsilon_{k'})} \cdot m_{\epsilon_{i'}+\epsilon_{k'}} + m_{\epsilon_k+\epsilon_{i'}}$$

and by 3.4(5) with i replaced by k', and with $b = \epsilon_{i'}$, we have

$$x^{(2\epsilon_{k'})} \cdot m_{\epsilon_{i'}+\epsilon_{k'}} = \lambda(\epsilon_k + \epsilon_{i'})_{k'} m_{\epsilon_k+\epsilon_{i'}},$$

whence

$$x^{(\epsilon_{k'}+\epsilon_i)} \cdot m_k = [1 - \lambda(\epsilon_k + \epsilon_{i'})_{k'}]m_{\epsilon_k+\epsilon_{i'}}. \tag{3.11.18}$$

Recall that $(i,j) \in I^-$, so either $1 \le i' < j \le r$ or $r < i \le j < n$. If $1 \le i' < j \le r$ (respectively, $r < i \le j < n$), then we can replace i by k in 3.4(2) (respectively, 3.4(3)) and then set $b = \epsilon_k + \epsilon_{i'}$ to obtain

$$x^{(\epsilon_k+\epsilon_j)} \cdot m_{\epsilon_k+\epsilon_{i'}} = \sigma(j')2^{\delta_{ij}} m_{\epsilon_{i'}+\epsilon_{j'}}. \tag{3.11.19}$$

If we instead replace i by i' and set $b = 2\epsilon_{i'}$ we obtain

$$m_{\epsilon_{i'}+\epsilon_{j'}} = \sigma(j')x^{(\epsilon_{i'}+\epsilon_j)} \cdot m_{2\epsilon_{i'}} \tag{3.11.20}$$

provided $j \neq i$. Now $m_{2\epsilon_{i'}} = 0$ by the preceding paragraph, so (3.11.20) is valid when $j = i$, and also $x^{(\epsilon_k+\epsilon_j)} \cdot m_{\epsilon_k+\epsilon_{i'}} = 0$ by (3.11.19) and (3.11.20). This, combined with (3.11.17) and (3.11.18), yields $x^{(\epsilon_i+\epsilon_j)} \cdot m_k = 0$, implying $m_k = 0$. But this contradicts the choice of k (see after (3.11.10)). Therefore, the assumption made before (3.11.10) that $m_a \neq 0$ must have been false. This completes the proof. □

3.12 Lemma. *Let $a \in A$. If $a \notin \{\epsilon_i \,|\, 1 \le i < n\} \cup \{0\}$, then $m_a = 0$.*

Proof. Assume $a \notin \{\epsilon_i \,|\, 1 \le i < n\} \cup \{0\}$. By 3.11, we may assume $a_i \in \{0,1\}$ for all $1 \le i < n$.

If $a_i = 1 = a_{j'}$ with $1 \le i,j \le r$, then $b = a + \epsilon_i - \epsilon_{j'}$ in 3.4(2) gives $m_a = 0$ (using 3.11), and if $a_i = 1 = a_j$ (respectively, $a_{i'} = 1 = a_{j'}$) with $1 \le i < j \le r$, then replacing j by j' in 3.4(3) and then putting $b = a + \epsilon_i - \epsilon_j$ (respectively, $b = a - \epsilon_{i'} + \epsilon_{j'}$) gives $m_a = 0$ (using 3.11).

Finally, if $a_n \neq 0$, then, using what we have just shown, we see that for any chosen $1 \le i < n$, putting $b = a + \epsilon_i - \epsilon_n$ in 3.4(7) yields $m_a = 0$. □

3.13. *Completion of proof of 1.3 when $\mathfrak{g} \in \{K, H\}$.*

Proof. First, we will require some formulas. Let $1 \le i,j \le r$ and $1 \le k < n$. From 3.4(1,2, and 3), respectively, we get

$$\begin{aligned} x^{(2\epsilon_i)} \cdot m_{\epsilon_k} &= -\delta_{ik} m_{\epsilon_{i'}}, \\ x^{(\epsilon_i+\epsilon_j)} \cdot m_{\epsilon_k} &= -\delta_{ik} m_{\epsilon_{j'}} - \delta_{jk} m_{\epsilon_{i'}}, \quad \text{if } i \neq j, \\ x^{(\epsilon_i+\epsilon_{j'})} \cdot m_{\epsilon_k} &= \delta_{ik} m_{\epsilon_j} - \delta_{j'k} m_{\epsilon_{i'}}, \quad \text{if } i < j. \end{aligned} \tag{3.13.1}$$

From 3.4(6) and 3.4(5), respectively, we get

$$\lambda(\epsilon_k)_i m_{\epsilon_k} = \begin{cases} \sigma(k)x^{(\epsilon_{i'}+\epsilon_{k'})} \cdot m_{\epsilon_{i'}} - \sigma(k)x^{(\epsilon_i+\epsilon_{k'})} \cdot m_{\epsilon_i}, & i \notin \{k,k'\}, \\ x^{(2\epsilon_{k'})} \cdot m_{\epsilon_{k'}}, & i \in \{k,k'\}. \end{cases} \tag{3.13.2}$$

Finally, if $\mathfrak{g} = K$, then 3.4(7) gives

$$\lambda(\epsilon_k)_n m_{\epsilon_k} = -\sigma(k) \sum_{\substack{1 \le l < n \\ l \notin \{k,k'\}}} x^{(\epsilon_{k'}+\epsilon_l)} \cdot m_{\epsilon_l} - 2\sigma(k)x^{(2\epsilon_{k'})} \cdot m_{\epsilon_{k'}} - \sigma(k)\lambda(\epsilon_k)_{k'} m_{\epsilon_k}. \tag{3.13.3}$$

Now suppose $m_a \neq 0$ for some $0 \neq a \in A$. We shall derive a contradiction. We begin by showing that M has a maximal vector of exceptional weight. By 3.12, we have $a = \epsilon_k$ for some $1 \leq k < n$. First suppose $r < k < n$. By replacing a if necessary, we may assume k is the least such integer for which $m_{\epsilon_k} \neq 0$. By (3.13.1), m_{ϵ_k} is a maximal vector. Then, by (3.13.2) and (3.13.3), with the aid of (3.13.1), we see that m_{ϵ_k} has the exceptional weight ω_k. Therefore, we may assume that $1 \leq k \leq r$ and that k is the greatest integer for which $m_{\epsilon_k} \neq 0$. By (3.13.1), (3.13.2), and (3.13.3), m_{ϵ_k} is a maximal vector of exceptional weight ω_{k-1}.

Checking the hypotheses of 1.3, we see that it must be the case that $\chi(\mathfrak{n}_0^-) \neq 0$. Therefore, $\chi(x^{(\epsilon_i+\epsilon_j)}) \neq 0$ for some $(i,j) \in I^-$, that is, with either $1 \leq i' < j \leq r$ or $r < i \leq j < n$.

We first show that $m_{\epsilon_{i'}} = 0$. If $j = i$, then $x^{(\epsilon_i+\epsilon_j)} \cdot m_{\epsilon_{i'}} = x^{(2\epsilon_i)} \cdot m_{\epsilon_{i'}} = 0$ by 3.4(1) with $l = 3$, $b = 0$, so $m_{\epsilon_{i'}} = 0$ (arguing as in the proof of 2.5). Hence, we may assume $j \neq i$. By 3.4(4) with i replaced by s, j replaced by t, and then $b = 0$, we have

$$\sigma(s)x^{(\epsilon_s+\epsilon_t)} \cdot m_{\epsilon_{s'}} = -\sigma(t)x^{(2\epsilon_s)} \cdot m_{\epsilon_{t'}} \tag{3.13.4}$$

for $0 \leq s,t < n$, $t \notin \{s, s'\}$. Then

$$\begin{aligned} x^{(\epsilon_i+\epsilon_j)}x^{(\epsilon_i+\epsilon_j)} \cdot m_{\epsilon_{i'}} &= \sigma(j)x^{(\epsilon_i+\epsilon_j)}x^{(2\epsilon_i)} \cdot m_{\epsilon_{j'}} = \sigma(j)x^{(2\epsilon_i)}x^{(\epsilon_i+\epsilon_j)} \cdot m_{\epsilon_{j'}} \\ &= x^{(2\epsilon_i)}x^{(2\epsilon_j)} \cdot m_{\epsilon_{i'}} = x^{(2\epsilon_j)}x^{(2\epsilon_i)} \cdot m_{\epsilon_{i'}} = 0, \end{aligned}$$

where we have used (3.13.4) for the first and third equalities, 3.1(4) for the second and fourth equalities, and 3.4(1) with $l = 3$, $b = 0$ again for the last equality. Thus, $m_{\epsilon_{i'}} = 0$ as desired.

Next, if $i' < l < n$, then $m_{\epsilon_l} = \sigma(l)x^{(\epsilon_{i'}+\epsilon_{l'})} \cdot m_{\epsilon_{i'}} = 0$ by (3.13.1) (using all three formulas in order for the cases $l = i$, $l > r$ with $l \neq i$, and $l \leq r$, respectively). Finally, let $1 \leq l < i'$. We have from 3.1(4)

$$\begin{aligned} 2^{\delta_{ij}}x^{(\epsilon_i+\epsilon_j)} \cdot m_{\epsilon_l} &= \langle x^{(\epsilon_i+\epsilon_l)}, x^{(\epsilon_j+\epsilon_{l'})}\rangle \cdot m_{\epsilon_l} \\ &= x^{(\epsilon_i+\epsilon_l)}x^{(\epsilon_j+\epsilon_{l'})} \cdot m_{\epsilon_l} - x^{(\epsilon_j+\epsilon_{l'})}x^{(\epsilon_i+\epsilon_l)} \cdot m_{\epsilon_l}. \end{aligned}$$

Using (3.13.4), we get $x^{(\epsilon_j+\epsilon_{l'})} \cdot m_{\epsilon_l} = \sigma(j)x^{(2\epsilon_{l'})} \cdot m_{\epsilon_{j'}} = 0$, since $m_{\epsilon_{j'}} = 0$ by the above argument and the fact that $j' \geq i'$. Also, $x^{(\epsilon_i+\epsilon_l)} \cdot m_{\epsilon_l} = m_{\epsilon_{i'}} = 0$ by (3.13.1). Therefore, $x^{(\epsilon_i+\epsilon_j)} \cdot m_{\epsilon_l} = 0$, implying $m_{\epsilon_l} = 0$.

We have shown that $m_{\epsilon_l} = 0$ for all $1 \leq l < n$. This contradicts our assumption that $m_{\epsilon_k} = m_a \neq 0$. It follows that $m_a = 0$ if $a \in A$ and $a \neq 0$, and we conclude that $v = 1 \otimes m_0$.

By 3.3, m_0 is a weight vector of weight $\lambda(0) = \lambda$. Since v is a maximal vector, $v \neq 0$, so $m_0 \neq 0$ as well. Finally 3.4(2 and 3) with $b = 0$ give $x^{(\epsilon_i+\epsilon_j)} \cdot m_0 = 0$ for $(i,j) \in I$. Thus m_0 is a maximal vector of weight λ. $\square$

4. Automorphisms and Conclusion.

In order to obtain our main results, we will find it convenient at times to assume that the character $\chi \in \mathfrak{g}^*$ is in a certain normal form, for instance, satisfying $\chi(\mathfrak{n}) = 0$. It turns out that, for our purposes, we may replace χ with a conjugate in normal form, where the conjugation is with respect to the action of the automorphism group of $\mathfrak{g}$ on $\mathfrak{g}^*$. (This is a technique employed by Friedlander and Parshall in their work on modular representations of the classical Lie algebras in [FP].) We begin this section by proving those conjugation results that will be required.

Let the notation be as in Section 1. We denote by $\operatorname{Aut}\mathfrak{g}$ the group of automorphisms of the Lie algebra $\mathfrak{g}$. Note that since $\mathfrak{g}$ is simple and hence centerless, each $\Phi \in \operatorname{Aut}\mathfrak{g}$ is automatically restricted, meaning $\Phi(D^{[p]}) = \Phi(D)^{[p]}$ for each $D \in \mathfrak{g}$. The group $\operatorname{Aut}\mathfrak{g}$ acts on the set $\mathfrak{g}^*$ according to the rule

$$\chi^{\Phi}(D) = \chi(\Phi(D))$$

($\chi \in \mathfrak{g}^*$, $\Phi \in \operatorname{Aut}\mathfrak{g}$, $D \in \mathfrak{g}$). If $\chi \in \mathfrak{g}^*$ and $\Phi \in \operatorname{Aut}\mathfrak{g}$, then, by the universal mapping property of reduced enveloping algebras, we easily get $u(\mathfrak{g}, \chi^{\Phi}) \cong u(\mathfrak{g}, \chi)$, so the theory of $\mathfrak{g}$-modules having character χ^{Φ} is the same as that of $\mathfrak{g}$-modules having character χ. It follows, for instance, that the number of isomorphism classes of simple $\mathfrak{g}$-modules having character $\chi \in \mathfrak{g}^*$ depends only on the conjugacy class of χ. However, we will see that in making other statements about the representation theory it is not so obvious that we can replace χ with a conjugate (see the application of 4.2 in the proof of 4.3, for instance).

We will require a rather detailed understanding of the structure of $\operatorname{Aut}\mathfrak{g}$. Fortunately, Wilson has provided such in [W]. The first result needed is that $\operatorname{Aut}\mathfrak{g} \cong \operatorname{Aut}^*\mathfrak{g} \ltimes \operatorname{Aut}_1\mathfrak{g}$, where $\operatorname{Aut}^*\mathfrak{g} = \{\Phi \in \operatorname{Aut}\mathfrak{g} \mid \Phi \text{ is homogeneous}\}$ and $\operatorname{Aut}_1\mathfrak{g} = \{\Phi \in \mathfrak{g} \mid (\Phi - 1_{\mathfrak{g}})(\mathfrak{g}_i) \subseteq \sum_{j>i}\mathfrak{g}_j \text{ for each } i\}$ (see [W, Theorem 2(a), p. 598]). Other of Wilson's findings will be called upon as required in the proofs below.

4.1 Theorem. *Let* $\chi \in \mathfrak{g}^*$.

(1) *If* $\Phi \in \operatorname{Aut}\mathfrak{g}$, *then* $\operatorname{ht}\chi^{\Phi} = \operatorname{ht}\chi$.
(2) *If* $\operatorname{ht}\chi \leq 1$, *then there exists* $\Phi \in \operatorname{Aut}^*\mathfrak{g}$ *such that* $\chi^{\Phi}(\mathfrak{n}) = 0$.
(3) *If* $\operatorname{ht}\chi = 1$, *then there exists* $\Phi \in \operatorname{Aut}\mathfrak{g}$ *such that* $\chi^{\Phi}(\mathfrak{g}^-) = 0$, *where* $\mathfrak{g}^- = \sum_{i<0}\mathfrak{g}_i$.

Proof. (1) Since $\operatorname{Aut}\mathfrak{g} = \operatorname{Aut}^*\mathfrak{g} \ltimes \operatorname{Aut}_1\mathfrak{g}$, it is enough to check the two cases $\Phi \in \operatorname{Aut}^*\mathfrak{g}$ and $\Phi \in \operatorname{Aut}_1\mathfrak{g}$, each of which is clear.

(2) If $\mathfrak{g} \in \{W, S\}$, put $C = \mathrm{SL}_n(F)$, $\mathfrak{c} = \mathfrak{sl}_n(F)$, while if $\mathfrak{g} \in \{K, H\}$, put $C = \mathrm{Sp}_{2r}(F)$, $\mathfrak{c} = \mathfrak{sp}_{2r}(F)$. Then $\mathfrak{c} = \operatorname{Lie} C$, and with the identification $\mathfrak{g}_0 \hookrightarrow \mathfrak{gl}_n(F)$ described at the beginning of Section 2 (respectively, Section 3), we have $\mathfrak{n}_0 \subseteq \mathfrak{c} \subseteq \mathfrak{g}_0$. Let $\psi = \chi|_{\mathfrak{c}}$. According to [KW, Theorem 4(iv), p. 140], there exists $g \in C$ satisfying $(g \cdot \psi)(\mathfrak{n}_0) = 0$

where $(g \cdot \psi)(x) = \psi(\mathrm{Ad}(g^{-1})(x))$. And by [W, Theorem 2(c and d), p. 598], there exists $\Phi \in \mathrm{Aut}^* \mathfrak{g}$ satisfying $\Phi(x) = g^{-1}xg$ for each $x \in \mathfrak{c}$. Then, $\chi^\Phi(\mathfrak{n}_0) = \chi(\Phi(\mathfrak{n}_0)) = \chi(g^{-1}\mathfrak{n}_0 g) = \psi(\mathrm{Ad}(g^{-1})(\mathfrak{n}_0)) = (g \cdot \psi)(\mathfrak{n}_0) = 0$. Finally, (1) says $\mathrm{ht}\,\chi^\Phi = \mathrm{ht}\,\chi \leq 1$, so $\chi^\Phi(\mathfrak{g}^1) = 0$. Thus, $\chi^\Phi(\mathfrak{n}) = 0$.

(3) First assume $\mathfrak{g} \in \{W, S\}$. We obtain a new grading on W by defining the kth homogeneous component to be $W_{[k]} = \sum F x^{(a)} D_j$, where the sum is over all $a \in A$, $1 \leq j \leq n$ for which $\sum_i i a_i - j = k$. Then, with the induced grading given by $\mathfrak{g}_{[k]} = \mathfrak{g} \cap W_{[k]}$, $\mathfrak{g}$ is a graded algebra and $\mathfrak{g}_0$ is a graded subalgebra.

Assume $\mathrm{ht}\,\chi = 1$. Then $\chi(\mathfrak{g}_0) \neq 0$, so there is a minimal t for which $\chi(\mathfrak{g}_0 \cap \mathfrak{g}_{[t]}) \neq 0$. We have $\chi(D) \neq 0$ for some $D \in \mathfrak{g}_{[t]}$ with either $D = h_n$ or $D = D_{ij}(x)$ with $x = x_a x_b$ for some $1 \leq i, j, a, b \leq n$.

If $\chi(\mathfrak{g}_{-1}) = 0$ there is nothing to show, so assume otherwise and let $1 \leq l \leq n$ be maximal for which $\chi(D_l) \neq 0$. Put

$$E = \begin{cases} x_l h_n, & \text{if } D = h_n \\ D_{ij}(x_l x), & \text{if } D = D_{ij}(x). \end{cases}$$

Then $E \in \mathfrak{g}_1 \cap \mathfrak{g}_{[t+l]}$.

Let $c \in F$. According to [W, Theorem 1], there exists $\Phi \in \mathrm{Aut}\,\mathfrak{g}$ such that $\Phi(D_k) - [cE, D_k] - D_k \in \mathfrak{g}^1$ for each k. Since $\mathrm{ht}\,\chi = 1$, we have $\chi(\mathfrak{g}^1) = 0$. Therefore,

$$\chi^\Phi(D_k) = \chi(\Phi(D_k)) = c\chi([E, D_k]) + \chi(D_k) \tag{4.1.1}$$

for each k.

Now $[E, D_k] \in \mathfrak{g}_0 \cap \mathfrak{g}_{[t+l-k]}$ so if $k > l$, then $\chi([E, D_k]) = 0$ (using minimality of t), which implies $\chi^\Phi(D_k) = \chi(D_k) = 0$. Also, if $D = h_n$, then

$$[E, D_l] = [x_l h_n, D_l] = [x_l x_n D_n, D_l] = -2^{\delta_{ln}} D,$$

while if $D = D_{ij}(x)$, then

$$[E, D_l] = -D_{ij}(D_l(x_l x)) = -D_{ij}(x) - D_{ij}(x_l(D_l x)) = -(1 + \delta_{la} + \delta_{lb})D$$

(the first equality from [SF, Lemma 3.2(4), p. 155]). In either case, the coefficient of D is nonzero. Since $\chi(D) \neq 0$ as well, we have $\chi([E, D_l]) \neq 0$. Therefore, we can let $c = -\chi([E, D_l])^{-1}\chi(D_l)$ in (4.1.1) to get $\chi^\Phi(D_l) = 0$.

We have shown that $\chi^\Phi(D_k) = 0$ for all $l \leq k \leq n$. Therefore, arguing by reverse induction on l we get the desired result.

Now assume $\mathfrak{g} \in \{K, H\}$. We obtain a new grading on $\mathfrak{g}$ by defining the kth homogeneous component to be $\mathfrak{g}_{[k]} = \sum F x^{(a)}$, where the sum is over all $a \in \hat{A}$ for which $\sum_{i=1}^r i(a_i - a_{i'}) = k$ (see discussion after 3.1 for notation). Then $\mathfrak{g}_0$ is a graded subalgebra.

Assume $\operatorname{ht}\chi = 1$ and let t be maximal with respect to the property $\chi(\mathfrak{g}_0 \cap \mathfrak{g}_{[t]}) \neq 0$. Then $\chi(D) \neq 0$ for some $D \in \mathfrak{g}_{[t]}$ of the form $D = x^{(a)}$ with $\|a\| = 0$. Furthermore, we may assume that if $a = \epsilon_n$, then $\chi(h_i) = 0$ for all $1 \leq i < n$, since $x^{(\epsilon_n)} = h_n$ and $\mathfrak{g}_0 \cap \mathfrak{g}_{[0]} = \mathfrak{h}$.

For $-r \leq k \leq r$, put

$$\tilde{k} = \begin{cases} k, & \text{if } 0 \leq k \leq r, \\ (-k)', & \text{if } -r \leq k < 0, \end{cases}$$

and define $y_k := x_{\tilde{k}}$. Note that $y_k \in \mathfrak{g}_{[k]}$ and $\mathfrak{g}^- = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} = \sum_{k=-r}^{r} F y_k$. Suppose $\chi(\mathfrak{g}^-) \neq 0$. Then $\chi(y_l) \neq 0$ for some maximal $-r \leq l \leq r$. Put $E = x^{(a+\epsilon_{\tilde{l}'})}$. Then $E \in \mathfrak{g}_{1+\delta_{l0}} \cap \mathfrak{g}_{[t-l]}$.

Let $c \in F$. As above, there exists $\Phi \in \operatorname{Aut}\mathfrak{g}$ such that

$$\chi^{\Phi}(y_k) = c\chi(\langle E, y_k\rangle) + \chi(y_k) \tag{4.1.2}$$

for each k.

The graded structure gives $\langle E, y_k\rangle \in \mathfrak{g}_{\delta_{l0}-\delta_{k0}} \cap \mathfrak{g}_{[t-l+k]}$. Explicitly we have from 3.1(1)

(4.1.3)

$$\langle E, y_k\rangle = \langle x^{(a+\epsilon_{\tilde{l}'})}, x_{\tilde{k}}\rangle = -\sigma(\tilde{k})x^{(a+\epsilon_{\tilde{l}'}-\epsilon_{\tilde{k}'})} - \left[(1-\delta_{0k})(a_{\tilde{k}} + \delta_{\tilde{k}\tilde{l}'}) + 1\right] x^{(a+\epsilon_{\tilde{l}'}+\epsilon_{\tilde{k}}-\epsilon_n)}.$$

Fix k with $l < k \leq r$. We claim that $\chi(\langle E, y_k\rangle) = 0$. If $l = 0$, then this is clear since $\chi(\mathfrak{g}_1) = 0$ by the assumption $\operatorname{ht}\chi = 1$. If $k = 0$, then $\langle E, y_k\rangle = -2x^{(a+\epsilon_{\tilde{l}'}-\epsilon_n)}$, which equals zero if $a \neq \epsilon_n$ and equals $-2y_{-l}$ otherwise, so we can use the definition of l since $-l > -k = 0 > l$. Finally, if $l \neq 0$, $k \neq 0$, then maximality of t gives the claim.

Next, (4.1.3) gives

$$\langle E, y_l\rangle = -\sigma(\tilde{l})D - \left[(1-\delta_{0l})a_{\tilde{l}} + 1\right] x^{(a+\epsilon_{\tilde{l}'}+\epsilon_{\tilde{l}}-\epsilon_n)}.$$

If $l = 0$, then the right-hand side is $-2D$. If $l \neq 0$ and the second term on the right is nonzero, then $a = \epsilon_n$, in which case this term is $-h_{\tilde{l}}$. In view of our assumptions on D, we conclude that $\chi(\langle E, y_l\rangle)$ is a nonzero multiple of $\chi(D)$ and is hence nonzero. Therefore, we can let $c = -\chi(\langle E, y_l\rangle)^{-1}\chi(y_l)$ and complete the proof as before. □

Let L be a restricted Lie algebra. Let $\Phi \in \operatorname{Aut}(L)$ and let M be an L-module. Denote by M^{Φ} the L-module having M as its underlying vector space and L-action given by $x \cdot m = \Phi(x)m$ ($x \in L$, $m \in M$), where the action on the right is the given one. Clearly, M^{Φ} is simple if and only if M is. Also, it is easy to check that if M has character χ, then M^{Φ} has character χ^{Φ}.

From the description before 4.1 of $\operatorname{Aut}\mathfrak{g}$, we see that any $\Phi \in \operatorname{Aut}\mathfrak{g}$ restricts to an automorphism of $\mathfrak{g}^0$ (respectively, $\mathfrak{g}^1$), which we continue to denote by Φ.

4.2 Proposition. *Let $\chi \in \mathfrak{g}^*$ and let $\Phi \in \operatorname{Aut} \mathfrak{g}$.*

(1) *If M is a $u(\mathfrak{g}^0, \chi)$-module, then $[Z^\chi(M)]^\Phi \cong Z^{\chi^\Phi}(M^\Phi)$.*

(2) *If $\operatorname{ht} \chi \leq 0$ and S is a simple $u(\mathfrak{g}_0, \chi)$-module, then $[Z^\chi(S)]^\Phi \cong Z^{\chi^\Phi}(S)$.*

Proof. (1) Let M be a $u(\mathfrak{g}^0, \chi)$-module. As noted above, $[Z^\chi(M)]^\Phi$ is a $u(\mathfrak{g}, \chi^\Phi)$-module. Its subspace $1 \otimes M$ is a $u(\mathfrak{g}^0, \chi^\Phi)$-submodule isomorphic to M^Φ. Moreover, a $u(\mathfrak{g}^0, \chi^\Phi)$-isomorphism $M^\Phi \to 1 \otimes M$ induces a $u(\mathfrak{g}, \chi^\Phi)$-homomorphism $f : Z^{\chi^\Phi}(M^\Phi) \to [Z^\chi(M)]^\Phi$, which is necessarily surjective since $1 \otimes M$ generates $[Z^\chi(M)]^\Phi$. Finally, both modules have dimension $p^k \dim_F M$, where $k = \sum_{i<0} \dim_F \mathfrak{g}_i$, so f is an isomorphism.

(2) Assume $\operatorname{ht} \chi \leq 0$ and let S be a simple module for $u(\mathfrak{g}_0, \chi) = u(\mathfrak{g}_0)$. By (1), it is enough to show that $S^\Phi \cong S$ as $\mathfrak{g}^0$-modules. Because $\mathfrak{g}^1$ acts trivially on S, we may assume that $\Phi \in \operatorname{Aut}^* \mathfrak{g}$. In particular, we need only show that $S^\Phi \cong S$ as $\mathfrak{g}_0$-modules. As in Wilson's paper [W], we view $\mathfrak{g}$ as a subalgebra of $W(n, \mathbf{1}) = \operatorname{Der}_F \mathfrak{A}(n, \mathbf{1})$ for an appropriate n. Set $\mathfrak{A} = \mathfrak{A}(n, \mathbf{1})$ and let $\operatorname{Aut}^* \mathfrak{A}$ denote the group of homogeneous automorphisms of $\mathfrak{A}$ (relative to Wilson's grading, which differs from ours when $\mathfrak{g} = K$). By [W, Theorem 2(b), p. 598], there exists $\varphi \in \operatorname{Aut}^* \mathfrak{A}$ satisfying $\Phi(x) = \varphi x \varphi^{-1}$ for all $x \in \mathfrak{g}$. If $\varphi_1 := \varphi|_{\mathfrak{A}_1} = c(\mathrm{id}_{\mathfrak{A}_1})$ for some $c \in F^\times$, then $\Phi(x) = x$ for each $x \in \mathfrak{g}_0$, implying $S^\Phi = S$. Therefore, by [W, Theorem 2(c and d), p. 598] we may assume $\varphi_1 \in G$, where G is $GL(\mathfrak{A}_1)$, $SL(\mathfrak{A}_1)$, $Sp(\mathfrak{A}_1') \times F^\times$ ($\mathfrak{A}_1' := \langle x_i \,|\, 1 \leq i < n \rangle$), or $Sp(\mathfrak{A}_1)$ according as $\mathfrak{g}$ is W, S, K, or H.

We view G as an F-group (scheme). It is reductive and $\operatorname{Lie} G = \mathfrak{g}_0$. Denote by G_1 the Frobenius kernel of G and let $\mathcal{F} : G_1\text{-}\mathbf{mod} \to u(\mathfrak{g}_0)\text{-}\mathbf{mod}$ and $\mathcal{G} : u(\mathfrak{g}_0)\text{-}\mathbf{mod} \to G_1\text{-}\mathbf{mod}$ denote the functors defining the equivalence of the indicated categories as described in [J, 8.6(2), using 8.5(b), p. 133]. Let M be a G_1-module, let $\alpha \in \operatorname{Aut} G_1$, and let M^α denote the G_1-module with G_1-action given by $g \cdot m = \alpha(g)m$ for any (commutative) F-algebra B, $g \in G_1(B)$, $m \in M \otimes B$, where the action on the right is the given one. Then $\mathcal{F}(M^\alpha) \cong [\mathcal{F}(M)]^{d\alpha}$, where $d\alpha$ is the differential of α. Indeed, any $x \in u(\mathfrak{g}_0)$ (identified with the algebra $M(G_1)$ of measures on G_1) acts on $\mathcal{F}(M^\alpha)$ as

$$\begin{aligned}(\mathrm{id}_{M^\alpha} \bar{\otimes} x) \circ \triangle_{M^\alpha} &= (\mathrm{id}_M \bar{\otimes} x) \circ (\mathrm{id}_M \otimes \alpha^*) \circ \triangle_M \\ &= [\mathrm{id}_M \bar{\otimes} (\alpha^*)^t(x)] \circ \triangle_M \\ &= (\mathrm{id}_M \bar{\otimes} d\alpha(x)) \circ \triangle_M,\end{aligned}$$

which is how x acts on $\mathcal{F}(M)^{d\alpha}$. This applies in particular to the choice $\alpha = \operatorname{Inn} \varphi_1 \in \operatorname{Aut} G_1$. Now, according to [J, 3.11, p. 220], if M is simple, then $M^\alpha \cong M$. Therefore,

$$S^\Phi = S^{\operatorname{Ad} \varphi_1} = S^{d\alpha} \cong \mathcal{F}(\mathcal{G}(S)^\alpha) \cong \mathcal{F}(\mathcal{G}(S)) \cong S,$$

where, for the first equality we have used that $(\operatorname{Ad} \varphi_1)(x) = \varphi_1 x \varphi_1^{-1}$ for any $x \in \mathfrak{g}_0$ (which is [J, 7.18(1), p. 126] applied to $M = \mathfrak{A}_1$). $\square$

For $\lambda \in \Lambda$, let $L_0(\lambda)$ be a simple $u(\mathfrak{g}_0)$-module having maximal vector of weight λ. Then $\{L_0(\lambda) \mid \lambda \in \Lambda\}$ is a complete set of pairwise nonisomorphic $u(\mathfrak{g}_0)$-modules.

4.3 Theorem. *Let $\chi \in \mathfrak{g}^*$ with $\operatorname{ht} \chi \leq 1$, and let S be a simple $u(\mathfrak{g}_0, \chi)$-module. If S is not $\mathfrak{g}_0$-isomorphic to any $L_0(\lambda)$ with $\lambda \in \Lambda_e$, then $Z^\chi(S)$ is simple. In particular, if $\operatorname{ht} \chi = 1$, then $Z^\chi(S)$ is simple.*

Proof. If $\operatorname{ht} \chi = 1$, then $\chi(\mathfrak{g}_0) \neq 0$, implying S is not restricted and hence not $\mathfrak{g}_0$-isomorphic to any $L_0(\lambda)$ with $\lambda \in \Lambda_e$. Therefore, it suffices to prove the first statement. In view of 4.2(1 and 2), we may replace χ with any convenient conjugate χ^Φ, $\Phi \in \operatorname{Aut} \mathfrak{g}$. Then by 4.1(2) we may assume $\chi(\mathfrak{n}) = 0$.

Assume S is not $\mathfrak{g}_0$-isomorphic to any $L_0(\mu)$ with $\mu \in \Lambda_e$. Let $v \in Z^\chi(S)$ be a maximal vector of weight, say, λ. By 1.1, it is enough to show that v generates $Z^\chi(S)$.

First assume $\chi(\mathfrak{n}_0^-) \neq 0$. By 1.3, we have $v = 1 \otimes m_0$ with $0 \neq m_0 \in S$. Since m_0 generates S, it follows that v generates $Z^\chi(S)$.

For the remainder of the proof, assume $\chi(\mathfrak{n}_0^-) = 0$. Suppose $\operatorname{ht} \chi \leq 0$. Then S is restricted and hence, by assumption, does not contain a maximal vector of exceptional weight. Therefore, $v = 1 \otimes m_0$ with $0 \neq m_0 \in S$ (by 1.3), and v generates $Z^\chi(S)$ as before.

Finally, suppose $\operatorname{ht} \chi = 1$. Then $\chi(h_i) \neq 0$ for some i. As pointed out before 1.1, $\lambda \in \Lambda^\chi$, so λ_i is a solution of $\lambda_i^p - \lambda_i = \chi(h_i)^p$. In particular, $\lambda_i \notin \mathbb{F}_p$. This shows that λ is not exceptional. Once again, 1.3 says $v = 1 \otimes m_0$ with $0 \neq m_0 \in S$ and the proof is complete. $\square$

The final result says essentially that the simple induced modules in the last theorem are pairwise nonisomorphic.

4.4 Theorem. *Let $\chi \in \mathfrak{g}^*$.*

(1) *Assume $\operatorname{ht} \chi \leq 0$. If $\lambda, \mu \in \Lambda \backslash \Lambda_e$ and $\lambda \neq \mu$, then $Z^\chi(L_0(\lambda)) \not\cong Z^\chi(L_0(\mu))$.*
(2) *Assume $\operatorname{ht} \chi = 1$. If S and S' are simple $u(\mathfrak{g}_0, \chi)$-modules and $S \not\cong S'$, then $Z^\chi(S) \not\cong Z^\chi(S')$.*

Proof. (1) Let $\lambda, \mu \in \Lambda \backslash \Lambda_e$ with $\lambda \neq \mu$. The $\mathfrak{g}_0$-module $L_0(\lambda)$ has a unique maximal vector m_0 up to scalar multiple and its weight is λ. Clearly $1 \otimes m_0 \in Z^\chi(L_0(\lambda))$ is a maximal vector of weight λ. Now $L_0(\lambda)$ has no maximal vector of exceptional weight, so 1.3 applies to show that every maximal vector of $Z^\chi(L_0(\lambda))$ is a multiple of $1 \otimes m_0$ and hence has weight λ. Similarly, $Z^\chi(L_0(\mu))$ has a maximal vector of uniquely determined weight μ, so the claim follows.

(2) By 4.2(1), we may replace χ with any conjugate χ^Φ, $\Phi \in \operatorname{Aut} \mathfrak{g}$. Then by 4.1(3), we may assume $\chi(\mathfrak{g}^-) = 0$. Then $u(\mathfrak{g}, \chi)$ is a graded algebra with grading induced by the grading on $\mathfrak{g}$ and, as such, satisfies the assumptions on the algebra A in [HN] (the

argument in Example 3 of Section 2 there carries over to $u(\mathfrak{g}, \chi)$). Then 3.2 of that paper establishes the claim. □

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